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Entailment, Transmission of Truth, and Minimality

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Abstract

Two concepts of single- and multiple-conclusion entailment, based on the idea of minimality, are introduced and studied. The analysis is performed at the propositional level. As for consistent sets of premises, ranges of the entailments defined, dubbed "strong", equal ranges of their classical counterparts. Yet, strong entailments are non-Tarskian. In particular, they are not monotone, but, at the same time, have some intuitively plausible properties which their standard counterparts lack. A proof-theoretic account of minimally inconsistent sets and thus, indirectly, of strong entailments is provided. Some applications of the introduced concepts, pertaining to belief revision and argument analysis, are discussed.

Keywords: entailment, multiple-conclusion entailment, non-monotonic logic, minimally inconsistent sets, contraction, argument analysis

1 Introduction

1.1 Single-Conclusion Entailment

The idea of *transmission of truth* underlies the intuitive concept of entailment. According to the idea, entailment is akin to an input-output device which, when fed with truth at the input, gives truth at the output. The input need not consist of truths, but *if* it does, it transforms into a true output. Similarly, *if* the premises are all true, any conclusion entailed by them must be true, although the truth of

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premises is not a necessary condition for entailment to hold. Or, to put it differently, the *hypothetical* truth of premises warrants the truth of an entailed conclusion.[1]

Logicians operate with well-formed formulas (*wffs* for short) of formalized languages and conceptualize entailment as a semantic relation between sets of wffs and single wffs. At the same time they tend to understand the "if" above in the sense of material conditional. Yet, since a material conditional with false antecedent is true irrespective of the logical value of the consequent, as a consequence one gets:

(I) *a set of wffs which cannot be simultaneously true, i.e. an inconsistent set, entails every wff.*

Moreover, a material conditional with true consequent is true irrespective of the logical value of the antecedent, and hence:

(II) *a logically valid wff is entailed by any set of wffs.*

Both (I) and (II) are a kind of by-products and we got accustomed to live with them. But (I) as well as (II) seem to contravene the intuitive idea of transmission of truth. To say that "truth is transmitted" seems to presuppose that it *can* occur at the input and that it *need not* occur at the output.

Another drawback of the received view is this. To say that the hypothetical truth of sentences in a set X warrants the truth of a sentence B seems to presuppose that the hypothetical truth of *all* the sentences in X contributes to the hypothetical truth of B. Entailment intuitively construed is a kind of semantic entrenchment of an entailed sentence in a set of sentences that entails it: a set of sentences X that entails a sentence B comprises neither less nor more sentences than those the hypothetical truth of which, jointly, warrants the truth of B. On the other hand, entailment defined in the usual way, by using, inter alia, the material "if", is monotone:

(M) *a wff B entailed by a set of wffs X is entailed by any superset of X as well*

and hence the wff B is also entailed by sets of wffs which contain elements that are irrelevant with regard to the transmission of truth and/or the semantic entrenchment effect(s): their hypothetical truth do not contribute in any way to the truth of B.

1.2 Multiple-Conclusion Entailment

The concept of entailment is sometimes generalized to the concept of multiple-conclusion entailment (*mc-entailment* for short). Mc-entailment is a semantic relation between sets of wffs, where an entailed set is allowed to contain more than one

[1] The latter statement can be explicated as: "If the truth-conditions of all the premises are met, an entailed conclusion is true as well."

element. The underlying idea is: an mc-entailed set must contain at least one true wff *if* the respective mc-entailing set consists of truths. Or, to put it differently, the hypothetical truth of all the wffs in an mc-entailing set warrants the existence of a true wff in the mc-entailed set.[2]

Mc-entailment can hold for trivial reasons: X mc-entails Y because X single-conclusion entails (*sc-entails* for short) at least one wff in Y. But mc-entailment can also hold non-trivially: it happens that a set of wffs, X, mc-entails a set of wffs, Y, although X does not sc-entail any wff in Y. For instance (taking Classical Propositional Logic as the basis), the truth of all the wffs in the set $X = \{p \to q \vee r, p\}$ warrants the existence of a true wff in the set $Y = \{q, r\}$, but neither q nor r is sc-entailed by X or, to put it differently, the hypothetical truth of the wffs in X guarantees that at least one of: q, r, is true, but warrants neither the truth of q nor the truth of r.

The concept of mc-entailment is more general than that of sc-entailment. One can always define sc-entailment as mc-entailment of a singleton set. However, it is not the case that mc-entailment can always be defined in terms of sc-entailment.[3]

One of the ways of thinking of entailed non-singleton *sets* is to construe them as items effectively *delimiting* search spaces: a set of wffs Y entailed by a set of wffs X is a minimal set that comprises wffs among which a truth must lie if the wffs in X are all true. "Minimal" means here "no proper subset of Y behaves analogously w.r.t. X." Another way of thinking about an entailed set is to construe it as characterizing the *relevant cases* to be considered, for if X mc-entails Y and each wff in Y sc-entails a wff B, the wff B is sc-entailed by X as well. However, the standard concept of mc-entailment is too broad to reflect the above ideas. This is due to the fact that mc-entailment is *right-monotone*.

(RM) *if a set of wffs X mc-entails a set of wffs, Y, then X mc-entails any superset of Y as well.*

Observe also that mc-entailment explicated by means of the material "if" suffers

[2]It is sometimes claimed that the concept of mc-entailment originates from [5] due to his introduction of sequents with sequences of wffs in the succedents. The semantic concept of mc-entailment was explicitly introduced in [4] under the heading "involution." Its syntactic counterpart, mc-consequence, was incorporated into the general theory of logical calculi by [19]. The first monograph devoted to mc-consequence and related concepts (multiple-conclusion calculus, multiple-conclusion rules, etc.) was [20].

[3]Philosophers tend to think that mc-entailment between sets X and Y is nothing more than sc-entailment of a disjunction of all the formulas in Y from a conjunction of all the formulas in X. However, this is not always so; whether this is right depends both on the richness of syntax (the presence/lack of infinite disjunctions and conjunctions) and semantics (the classical/nonclassical meanings of disjunction and conjunction). For details and counterexamples see [21].

similar drawbacks to those of sc-entailment explicated in this way:

(I') *any set of wffs is mc-entailed by an inconsistent set of wffs,* and

(II') *a set of wffs that contains a logically valid wff is mc-entailed by any set of wffs.*

Moreover, mc-entailment is *left-monotone*, that is:

(LM) *a set of wffs Y which is mc-entailed by a set of wffs X is also mc-entailed by any superset of X.*

Hence there exist mc-entailing sets of wffs which contain, inter alia, wffs that are semantically irrelevant to the corresponding mc-entailed sets. For example, $\{s, p \to q \vee r, p\}$ mc-entails $\{q, r\}$, while the hypothetical truth of s is completely irrelevant to the occurrence of truth in $\{q, r\}$.

1.3 Aims

In this paper we introduce and examine a concept of multiple-conclusion entailment, which we dub "strong multiple-conclusion entailment." Formally, strong mc-entailment is a subrelation of mc-entailment. We define strong mc-entailment in a way which allows us to avoid the drawbacks (I') and (II') indicated above. Moreover, strong mc-entailment is neither left-monotone nor right-monotone. As a by-product one gets a concept of single-conclusion entailment which, in turn, is free of the drawbacks pointed out at the beginning of this paper. We coin the concept "strong single-conclusion entailment." This concept of sc-entailment is analysed in the paper as well.

For simplicity, the analysis is pursued at the propositional level.

The paper is organized as follows.

Section 2 introduces the logical apparatus needed.

Section 3 is devoted to strong mc-entailment. Its definition is proposed and the adequacy issue is addressed therein. The consecutive subsections include theorems and corollaries characterizing basic properties of strong mc-entailment. In particular, it is shown that strong mc-entailment between non-empty sets of wffs involves only such sets which comprise contingent (i.e. neither valid nor inconsistent) wffs. It occurs that, as long as the underlying logic has the compactness property, strong mc-entailment holds only between finite sets of wffs. Strong mc-entailment exhibits the variable-sharing property. The relation between strong mc-entailment and minimally inconsistent sets is examined. A deduction theorem for strong mc-entailment is provided.

An analysis of strong sc-entailment is presented in Section 4. We define it as strong mc-entailment of a singleton set. Basic properties of strong sc-entailment are analysed. It is shown that the relation between strong mc- and sc-entailments does not fit the "a conjunction yields a disjunction" pattern.

In Section 5 we compare strong sc- and mc-entailment with their classical counterparts. We show that classical sc-entailment from a consistent set of wffs boils down to strong sc-entailment from a finite subset of the set: any wff classically sc-entailed by a consistent set of wffs is also strongly sc-entailed by a finite subset of the set. An analogous result for strong mc-entailment is proven as well. Section 5 includes also some comparative remarks on strong entailments and accounts of entailment proposed in relevance logics and connective logics.

Section 6 provides a proof-theoretic account of minimally inconsistent sets and thus, indirectly, of strong mc- and sc-entailments. Proofs of soundness and completeness of the proposed calculus are given in the Appendix.

Section 7 is devoted to some conceptual applications of the results presented in the previous sections.

Section 8 discusses the issue of transferability of the results concerning the classical propositional case to the first-order level and, very briefly, to the case of non-classical logics.

Some, but not all, of the results presented in this paper were already made available in the research report [28].

2 The Logical Basis

We remain at the propositional level, and we consider the case of Classical Propositional Logic (hereafter: CPL). We assume that CPL is expressed in a language characterized as follows.

The vocabulary of the language comprises a countably infinite set $\mathtt{Var}$ of propositional variables, the connectives: $\neg, \vee, \wedge, \rightarrow$, and brackets. The set $\mathtt{Form}$ of *well-formed formulas* (wffs) of the language is the smallest set that includes $\mathtt{Var}$ and satisfies the following conditions: (1) if $A \in \mathtt{Form}$, then '$\neg A' \in \mathtt{Form}$; (2) if $A, B \in \mathtt{Form}$, then '$(A \otimes B)$' $\in \mathtt{Form}$, where $\otimes$ is any of the connectives: $\vee, \wedge, \rightarrow$. We adopt the usual conventions concerning omitting brackets. We use A, B, C, D, with subscripts when needed, as metalanguage variables for wffs, and X, Y, W, Z, with or without subscripts or superscripts, as metalanguage variables for sets of wffs. The letters p, q, r, s, t are exemplary elements of $\mathtt{Var}$.

By a proper superset of a set of wffs X we mean a set of wffs Z such that X is a proper subset of Z. For the sake of brevity, we adopt the following notational conventions:

- we write X, Y instead of $X \cup Y$,
- X, A abbreviates $X \cup \{A\}$,
- $X_{\ominus A}$ abbreviates $X \setminus \{A\}$.

These conventions will be applied as long as there is no risk of a misunderstanding.

The inscriptions $\bigwedge X$ and $\bigvee Y$ refer to a conjunction of all the wffs in a non-empty and finite set of wffs X and to a disjunction of all the wffs in X, respectively. If X is a singleton set, $\{A\}$, then $\bigwedge X = \bigvee X = A$.

Let $\mathbf{1}$ stand for truth and $\mathbf{0}$ for falsity. A CPL-*valuation* is a function $v : \texttt{Form} \mapsto \{\mathbf{1}, \mathbf{0}\}$ satisfying the following conditions: (a) $v(\neg A) = \mathbf{1}$ iff $v(A) = \mathbf{0}$; (b) $v(A \vee B) = \mathbf{1}$ iff $v(A) = \mathbf{1}$ or $v(B) = \mathbf{1}$; (c) $v(A \wedge B) = \mathbf{1}$ iff $v(A) = \mathbf{1}$ and $v(B) = \mathbf{1}$; (d) $v(A \to B) = \mathbf{1}$ iff $v(A) = \mathbf{0}$ or $v(B) = \mathbf{1}$. Remark that the domain of v includes $\texttt{Var}$.

For brevity, in what follows we will be omitting references to CPL. Unless otherwise stated, the semantic relations analysed are supposed to hold between sets of CPL-wffs, or sets of CPL-wffs and single CPL-wffs. By valuations we will mean CPL-valuations.

We define:

Definition 1 (Sc-entailment). *$X \models A$ iff for each valuation v:*

- *if $v(B) = \mathbf{1}$ for every $B \in X$, then $v(A) = \mathbf{1}$.*

Wffs A and B are *logically equivalent* iff $A \models B$ and $B \models A$. Sets of wffs, X and Y, are logically equivalent iff they have exactly the same models, where a model of a set of wffs is a valuation which makes true all the wffs in the set.

Definition 2 (Mc-entailment). *$X \mathrel{||}\!\!= Y$ iff for each valuation v:*

- *if $v(B) = \mathbf{1}$ for every $B \in X$, then $v(A) = \mathbf{1}$ for at least one $A \in Y$.*

Definition 3 (Consistency, inconsistency, validity, and contingence). *A set of wffs X is* consistent *iff there exists a valuation v such that for each $A \in X$, $v(A) = \mathbf{1}$; otherwise X is* inconsistent*. A wff B is:*

1. consistent *iff the singleton set $\{B\}$ is consistent,*
2. inconsistent *iff the singleton set $\{B\}$ is inconsistent,*

3. valid *iff for each valuation v, $v(B) = \mathbf{1}$,*

4. contingent *iff B is neither inconsistent nor valid.*

Remark 1. Consistent wffs construed in the above manner are often called satisfiable wffs. The category of contingent wffs comprises wffs which are satisfiable, but not valid.

3 Strong Multiple-Conclusion Entailment

3.1 Definition and the Adequacy Issue

We use $\Vert\!\!\prec$ as the symbol for strong mc-entailment, and we define the relation as follows:[4]

Definition 4 (Strong mc-entailment)**.** *$X \Vert\!\!\prec Y$ iff*

1. $X \Vert\!\!= Y$, and

2. for each $A \in X : X_{\ominus A} \Vert\!\!\not= Y$, and

3. for each $B \in Y : X \Vert\!\!\not= Y_{\ominus B}$.

The consecutive clauses of the above definition express the following intuitions: the hypothetical truth of all the wffs in X warrants the existence of at least one true wff in Y, yet the warranty disappears as X decreases or Y decreases. In other words, X and Y are minimal sets under the warranty provided by the clause 1.

Here are simple examples:

$$\{p\} \Vert\!\!\prec \{p\} \tag{1}$$

$$\{p, p \to q\} \Vert\!\!\prec \{q\} \tag{2}$$

$$\{p \vee q, \neg p\} \Vert\!\!\prec \{q\} \tag{3}$$

$$\emptyset \Vert\!\!\prec \{p, \neg p\} \tag{4}$$

$$\{p, \neg p\} \Vert\!\!\prec \emptyset \tag{5}$$

$$\emptyset \Vert\!\!\prec \{p \vee \neg p\} \tag{6}$$

$$\{\neg p, \neg q, p \vee q\} \Vert\!\!\prec \emptyset \tag{7}$$

$$\emptyset \Vert\!\!\prec \{p, q, \neg(p \vee q)\} \tag{8}$$

[4]Recall that $X_{\ominus A}$ abbreviates $X \setminus \{A\}$, and similarly for Y.

$$\{p \vee q\} \parallel\!\prec \{p, q\} \tag{9}$$
$$\{p \vee q\} \parallel\!\prec \{p \wedge q, p \wedge \neg q, \neg p \wedge q\} \tag{10}$$
$$\{p \wedge q \to r, \neg r\} \parallel\!\prec \{\neg p, \neg q\} \tag{11}$$
$$\{p \wedge (q \vee r)\} \parallel\!\prec \{p \wedge q, p \wedge r\} \tag{12}$$
$$\{p \vee (q \vee r)\} \parallel\!\prec \{p \vee q, p \vee r\} \tag{13}$$
$$\{\neg(p \wedge (q \wedge r))\} \parallel\!\prec \{\neg p \vee \neg q, \neg p \vee \neg r\} \tag{14}$$
$$\{p \vee (q \vee r)\} \parallel\!\prec \{(p \vee q) \wedge (p \vee r), q \vee r\} \tag{15}$$

Note that $\emptyset \parallel\!\not\prec \emptyset$, as $\emptyset \parallel\!\not\models \emptyset$.

Since the empty set has no proper subsets, and each proper subset of a non-empty set is included in a maximal proper subset of the set, it is clear that the following is true:

Corollary 1. *$X \parallel\!\prec Y$ iff $X \parallel\!\models Y$ and the following conditions hold:*

1. *there is no proper subset Z of X such that $Z \parallel\!\models Y$,*
2. *there is no proper subset W of Y such that $X \parallel\!\models W$.*

Due to the monotonicity of "standard" mc-entailment, $\parallel\!\models$, we have:

Corollary 2. *If $X \parallel\!\prec Y$, then:*

1. *$Z \parallel\!\not\prec Y$, where Z is either a proper subset or a proper superset of X,*
2. *$X \parallel\!\not\prec W$, where W is either a proper subset or a proper superset of Y.*

Thus strong mc-entailment, $\parallel\!\prec$, is neither left-monotone nor right-monotone. The examples presented below witness this:

$$\{p, p \to q \vee r\} \parallel\!\prec \{q, r\} \tag{16}$$
$$\{p, p \to q \vee r, \neg q\} \parallel\!\not\prec \{q, r\} \tag{17}$$
$$\{p, p \to q \vee r\} \parallel\!\not\prec \{q, r, q \vee r\} \tag{18}$$

Observe that the following are true:[5]

$$\{p, \neg p\} \parallel\!\not\prec \{q\} \tag{19}$$
$$\{p\} \parallel\!\not\prec \{p \vee \neg p\} \tag{20}$$

Thus it is neither the case that any inconsistent set of wffs strongly mc-entails any set of wffs nor it is the case that a set which contains a valid wff is strongly mc-entailed by any set of wffs. Hence strong mc-entailment is free of the drawbacks (I') and (II') pointed out in section 1.2.

[5] As for (19), $\{q\} \setminus \{q\} = \emptyset$, but we have $\{p, \neg p\} \parallel\!\models \emptyset$. In the case of (20) we have $\emptyset \parallel\!\models \{p \vee \neg p\}$.

3.1.1 Strong Mc-entailment, Perfect Validity, and Tennant's Entailments

Corollary 1 yields that our concept of strong mc-entailment is akin to (but not identical with) the concept of perfectly valid sequent introduced in [25], p. 185.

Assume for a moment that sequents are simply pairs of sets of wffs. A proper subsequent of a sequent $X : Y$ is a sequent resulting from it by removing at least one wff from X or from Y. Tennant's definition of validity of a sequent $X : Y$ amounts to the presence of mc-entailment of Y from X. A sequent $X : Y$ is *perfectly valid* iff $X : Y$ is valid and no proper subsequent of $X : Y$ is valid. Thus, by Corollary 1, a sequent $X : Y$ is perfectly valid iff $X \Vdash\!\!\prec Y$ holds.[6]

However, perfect validity performs an auxiliary role in [25]. The central concept is that of sequent being an entailment. A sequent $X : Y$ is an *entailment* just in case $X : Y$ has a perfectly valid suprasequent. A sequent $Z : W$ is a suprasequent of the sequent $X : Y$ iff for some substitution s, $s(Z) = X$ and $s(W) = Y$. Tennant builds a sequent calculus which is sound and complete w.r.t. entailments construed in the above manner. A proof-theoretic account of perfectly valid sequents is also given by means of the so-called perfect proofs.

In this paper we will concentrate on a semantic analysis of strong mc-entailment or, if you prefer, perfect validity. A proof-theoretic account of strong mc-entailment, different from that offered by Tennant for perfect validity, will be also provided.

3.2 Basic Properties of Strong Mc-entailment

Let us first note:

Corollary 3. *Let A, B be logically equivalent wffs.*

1. *If $A \in X$ and $X \Vdash\!\!\prec Y$, then $X_{\ominus A} \cup \{B\} \Vdash\!\!\prec Y$.*
2. *If $A \in Y$ and $X \Vdash\!\!\prec Y$, then $X \Vdash\!\!\prec Y_{\ominus A} \cup \{B\}$.*

Thus logically equivalent wffs are replaceable in the context of strong mc-entailment. Needless to say, replaceability may fail for logical equivalence of sets of wffs. This

[6] But if the concept of proper subsequent is to be understood differently (i.e. $X' : Y'$ is a proper subsequent of $X : Y$ just in case $X' \subsetneqq X$ or $Y' \subsetneqq Y$, one needs the condition:

$$X' \cup Y' \subseteq X \cup Y$$

in order to pass from perfect validity to strong mc-entailment. Tennant does not provide an explicit definition of the notion of proper subsequent used.

is not surprising, as strong mc-entailment is a “hybrid” notion, defined in terms of semantic as well as set-theoretic clauses.[7]

Corollary 4. $\{A\} \Vert\!\prec \{A\}$ *iff* A *is contingent.*

Proof. Clearly, $\{A\} \Vert\!\models \{A\}$, and $\{A\}_{\ominus A} = \emptyset$. On the other hand, A is not valid iff $\emptyset \Vert\!\not\models \{A\}$, and A is not inconsistent iff $\{A\} \Vert\!\not\models \emptyset$. □

However, the overlap/reflexivity condition is not satisfied in the case of non-singleton sets.

Corollary 5. *If* X *has at least two elements, then* $X \Vert\!\not\prec X$*.*

Proof. Suppose otherwise. It follows that $X_{\ominus A} \Vert\!\not\models X$, where $A \in X$. But, as X has at least two elements, it holds that $X_{\ominus A} \cap X \neq \emptyset$ and hence $X_{\ominus A} \Vert\!\models X$. A contradiction. □

Let us now prove

Corollary 6. *If* $X \Vert\!\prec Y$ *and* X *is inconsistent, then* $Y = \emptyset$*.*

Proof. Let $X \Vert\!\prec Y$. Thus $X \Vert\!\models Y$. Assume that X is inconsistent. Suppose that $Y \neq \emptyset$. Thus $\emptyset$ is a proper subset of Y. However, $X \Vert\!\models \emptyset$ (since X is inconsistent) and hence $X \Vert\!\not\prec Y$ due to Corollary 1. So $Y = \emptyset$. □

Thus an inconsistent set strongly mc-entails, if any, only the empty set. If any, since there are inconsistent sets that do not strongly mc-entail even the empty set. For instance, the set $\{p \wedge \neg p, p\}$ does not strongly entail the empty set because we still have $\{p \wedge \neg p\} \Vert\!\models \emptyset$. As we will see, only minimally inconsistent sets strongly mc-entail the empty set.

Remark 2. There exist strongly mc-entailed inconsistent sets of wffs. Examples (4) and (8) presented above support this claim. Here are examples which do not involve the empty set:

$$\{p\} \Vert\!\prec \{p \wedge q, p \wedge \neg q\} \tag{21}$$

$$\{\neg(p \wedge q), p \vee q\} \Vert\!\prec \{p \wedge \neg q, \neg p \wedge q\} \tag{22}$$

[7] Such a solution has obvious vices, but also some virtues; see sections 4.2.3 and 5.1 below.

3.2.1 Contingent, Valid, and Inconsistent Wffs

Interestingly enough, strong mc-entailment between non-empty sets of wffs involves sets which comprise contingent wffs only. The following holds:

Theorem 1 (Contingency). *Let $X \Vert\!\prec Y$. If $X \neq \emptyset$ and $Y \neq \emptyset$, then each wff in $X \cup Y$ is contingent.*

Proof. Assume that $X \Vert\!\prec Y$, where X and Y are non-empty sets.

Suppose that X contains a valid wff, say, A. It follows that $X_{\ominus A} \Vert\!= Y$ and therefore $X \Vert\!\not\prec Y$. Now suppose that X contains an inconsistent wff. Hence X is an inconsistent set. But $Y \neq \emptyset$. Thus, by Corollary 6, $X \Vert\!\not\prec Y$, which contradicts the assumption.

Therefore X contains contingent wffs only.

Suppose that a valid wff, say, A, belongs to Y. By assumption, $X \neq \emptyset$, so $\emptyset$ is a proper subset of X. Suppose that $Y = \{A\}$. Clearly, $\emptyset \Vert\!= \{A\}$ due to the validity of A. Hence $X \Vert\!\not\prec \{A\}$. Now suppose that $Y \neq \{A\}$. As $Y \neq \emptyset$, it follows that $\{A\}$ is a proper subset of Y which, however, is mc-entailed by X since A is valid. Thus $X \Vert\!\not\prec Y$. Therefore no valid wff belongs to Y.

Finally, suppose that an inconsistent wff, B, belongs to Y. In this case $X \Vert\!= Y$ yields $X \Vert\!= Y_{\ominus B}$. As Y is, by assumption, non-empty, $Y_{\ominus B}$ is a proper subset of Y. It follows that $X \Vert\!\not\prec Y$. We arrive at a contradiction. Thus no wff in Y is inconsistent.

Therefore $X \cup Y$ contains contingent wffs only. □

What if either X or Y is empty? The answer is provided by:

Corollary 7.

1. *If $\emptyset \Vert\!\prec Y$, then either Y is a singleton set containing a valid wff, or Y is a non-singleton set comprising only contingent wffs.*
2. *If $X \Vert\!\prec \emptyset$, then either X is a singleton set containing an inconsistent wff, or X is a non-singleton set comprising only contingent wffs.*

Proof. If $\emptyset \Vert\!\prec Y$, then $Y \neq \emptyset$. Assume that Y is a singleton set, $\{C\}$. Since $\emptyset \Vert\!\prec \{C\}$ presupposes $\emptyset \Vert\!= \{C\}$, it follows that C is a valid wff. Assume that Y is a non-singleton set. Suppose that Y contains a non-contingent wff, say, B. If B is valid, then $\emptyset \Vert\!= \{B\}$ and hence $\emptyset \Vert\!\not\prec Y$. The situation is analogous when B is an inconsistent wff; in this case we would have $\emptyset \Vert\!= Y_{\ominus B}$.

If $X \Vert\!\prec \emptyset$, then $X \neq \emptyset$. Assume that X is a singleton set, $\{C\}$. Thus $\{C\} \Vert\!= \emptyset$ and hence C is an inconsistent wff. Assume that X is a non-singleton set. Suppose that X contains a non-contingent wff, say, A. Clearly, $X_{\ominus A}$ is a proper subset of X and so is $\{A\}$. Assume that A is valid. Thus $X_{\ominus A} \Vert\!\models \emptyset$ and hence $X \Vert\!\prec \emptyset$ does not hold. Now assume that A is inconsistent. Thus $\{A\} \Vert\!\models \emptyset$ and hence, again, $X \Vert\!\prec \emptyset$ is not the case. Therefore each wff in X is contingent provided that X is a non-singleton set. □

As for strong mc-entailment, non-contingent wffs come into play in two exceptional situations only.

Theorem 2. *Let $X \Vert\!\prec Y$.*

1. *If C is valid, then: $C \in X \cup Y$ iff $X = \emptyset$ and $Y = \{C\}$.*
2. *If C is inconsistent, then: $C \in X \cup Y$ iff $X = \{C\}$ and $Y = \emptyset$.*

Proof. Let C be a valid wff. Assume that $X \Vert\!\prec Y$ and $C \in X \cup Y$.

Suppose that $C \in X$. Hence $X \neq \emptyset$ and $X_{\ominus C}$ is a proper subset of X. If C is valid, then whatever is mc-entailed by X is also mc-entailed by $X_{\ominus C}$. So $X \Vert\!\not\prec Y$. We arrive at a contradiction. Therefore $C \notin X$ and thus $C \in Y$.

Suppose that $X \neq \emptyset$. Thus $\emptyset$ is a proper subset of X. Since C is valid and $C \in Y$, we have $\emptyset \Vert\!\models Y$. It follows that $X \Vert\!\not\prec Y$, contrary to the assumption. Therefore $X = \emptyset$. As $C \in Y$, it follows that Y does not comprise contingent wffs only. Hence $Y = \{C\}$ due to Corollary 7.

Needless to say, if $Y = \{C\}$, then $C \in X \cup Y$.

The proof of (*2*) goes along similar lines. □

According to Theorem 2, valid wffs can occur as elements of strongly mc-entailed sets, but these sets are always singleton sets which, moreover, are strongly mc-entailed only by the empty set. Similarly, if an inconsistent wff belongs to a strongly mc-entailing set, it is the only element of this set and the respective strongly mc-entailed set is empty. Moreover, valid wffs never occur in strongly mc-entailing sets, and inconsistent wffs never occur in strongly mc-entailed sets.

3.2.2 Strict Finiteness and Variable Sharing

We are dealing here with CPL, in which mc-entailment has the following properties:

(lf) *If $X \Vert\!\models Y$, then $X_1 \Vert\!\models Y$ for some finite subset X_1 of X.*

(rf) *If $X \Vert\!\models Y$, then $X \Vert\!\models Y_1$ for some finite subset Y_1 of Y.*

As for CPL (and other logics in which mc-entailment fulfils the above conditions), strong mc-entailment is strictly finitistic in the sense explained by:

Theorem 3 (Strict finiteness). *If $X \Vdash\!\!\prec Y$, then X and Y are finite sets.*

Proof. Let $X \Vdash\!\!\prec Y$.

Suppose that X is an infinite set. By Corollary 1, it follows that there is no finite subset of X which mc-entails Y. Hence $X \not\Vdash Y$ due to condition (lf). But $X \Vdash\!\!\prec Y$ yields $X \Vdash Y$. So X is a finite set. Now suppose that Y is an infinite set. Hence, by Corollary 1, no finite subset of Y is mc-entailed by X. Thus $X \not\Vdash Y$ due to condition (rf). It follows that $X \Vdash\!\!\prec Y$ does not hold, contrary to the assumption. So Y is a finite set as well. $\square$

Our next theorem is strongly dependent on the fact that we consider here propositional formulas.

Notation. By $\mathtt{Var}(A)$ we designate the set of all the propositional variables that occur in a wff A. $\mathtt{Var}(X)$ designates the set of all the propositional variables that occur in the wffs which belong to a set of wffs X.

Theorem 4 (Variable sharing). *Let $X \Vdash\!\!\prec Y$. If X and Y are non-empty sets, then* $\mathtt{Var}(X) \cap \mathtt{Var}(Y) \neq \emptyset$.

Proof. Let $X \Vdash\!\!\prec Y$, where $X \neq \emptyset$ and $Y \neq \emptyset$.

If $X \neq \emptyset$, then, by Corollary 2, $\emptyset \not\Vdash Y$. By assumption, $Y \neq \emptyset$. So there exists a valuation, say, v^*, such that $v^*(B) = \mathbf{0}$ for any $B \in Y$. By Corollary 6, X is consistent. Hence there exists a valuation v such that $v(A) = \mathbf{1}$ for every $A \in X$.

Suppose that $\mathtt{Var}(X) \cap \mathtt{Var}(Y) = \emptyset$. Let v^+ be a valuation such that: (a) $v^+(p_i) = v^*(p_i)$ if $p_i \in \mathtt{Var}(Y)$, (b) otherwise $v^+(p_i) = v(p_i)$. As $\mathtt{Var}(X) \cap \mathtt{Var}(Y) = \emptyset$, we have $v^+(A) = \mathbf{1}$ for every $A \in X$. On the other hand, $v^+(B) = \mathbf{0}$ for each $B \in Y$. Hence $X \not\Vdash Y$ and therefore $X \Vdash\!\!\not\prec Y$. We arrive at a contradiction. $\square$

So when strong mc-entailment between X and Y holds, the wffs in X share propositional variable(s) with the wffs in Y. However, Theorem 4 cannot be strengthened to the effect that $\mathtt{Var}(Y) \subseteq \mathtt{Var}(X)$ would be the case. Similarly, $\mathtt{Var}(X) \subseteq \mathtt{Var}(Y)$ does not generally hold.[8]

3.2.3 Partial Reduction to Minimally Inconsistent Sets

As long as a logic operating with the classical negation is concerned, there exist simple links between strong mc-entailment and minimally inconsistent sets:[9]

[8] For instance, we have $\{p \vee q\} \Vdash\!\!\prec \{p, r \to q\}$ as well as $\{p \wedge q\} \Vdash\!\!\prec \{p\}$.

[9] The concept of minimally inconsistent set has found natural applications is many areas, from philosophy of science (cf., e.g., [9]) to theoretical computer science, AI, and logic (see, e.g., [12],

Definition 5 (Minimally inconsistent set; MI-set). *A set of wffs X is* minimally inconsistent *iff X is inconsistent, but each proper subset of X is consistent.*

For brevity, we will be referring to minimally inconsistent sets as to MI-*sets*.

Note that $\emptyset$ is not a MI-set. Singleton MI-sets have inconsistent wffs as the (only) elements. Here are examples of non-singleton MI-sets:

$$\{p, \neg p\} \tag{23}$$

$$\{p \vee q, \neg p, \neg q\} \tag{24}$$

$$\{p \to q, p, \neg q\} \tag{25}$$

$$\{p \to q \vee r, p, \neg q, \neg r\} \tag{26}$$

$$\{p \to q, q \to r, \neg(p \to r)\} \tag{27}$$

Clearly, the following holds:

Corollary 8. *X is a* MI-*set iff X is inconsistent and for each $A \in X$, the set $X_{\ominus A}$ is consistent.*

Remark 3. As for CPL, any MI-set is finite. This is due to the fact that the following *compactness claim* holds for CPL:

(♣) *for each set of wffs Z: the set Z is consistent iff each finite subset of Z is consistent.*

However, there are logics for which the analogues of (♣) do not hold and thus finiteness is not a property of MI-sets in general.[10]

Notation. For brevity, we put:

$$\neg Y =_{df} \{\neg A : A \in Y\}$$

In the case of CPL, strong mc-entailment and MI-sets are linked in the following way:

Theorem 5. *$X \Vert\!\prec Y$ iff $X \cap \neg Y = \emptyset$ and $X, \neg Y$ is a* MI-*set.*

[3], [16]). Minimally inconsistent sets are also called *minimal unsatisfiable (sub)sets* or *unsatisfiable cores*.

[10]For example, in a logic that validates the ω-rule, a set of the form $\{\exists x Px\} \cup \{\neg Pa : a \in \mathsf{T}\}$, where P is a predicate and T is a (countably infinite) set of all closed terms of the language, is an infinite MI-set.

Proof. ($\Rightarrow$) Let $X \Vdash\!\prec Y$. Suppose that $X \cap \neg Y \neq \emptyset$. Let $A \in X \cap \neg Y$. Thus $A = \neg B$ for some $B \in Y$. Let $Y^* = Y_{\ominus B}$. From $X \Vdash\!\prec Y$ we get $X \Vdash\!= Y^*, B$. Therefore $X, \neg B \Vdash\!= Y^*$, that is, $X, A \Vdash\!= Y^*$. But $X, A = X$, since $A \in X$. Hence X mc-entails the proper subset Y^* of Y. It follows that $X \Vdash\!\not\prec Y$. We arrive at a contradiction. Therefore $X \cap \neg Y = \emptyset$.

If $X \Vdash\!\prec Y$, then $X \Vdash\!= Y$ and thus the set $X, \neg Y$ is inconsistent. Let us designate the set $X, \neg Y$ by Z.

If $A \in Z$, then $A \in X$ or $A \in \neg Y$.

Assume that $A \in X$. By the clause 2 of Definition 4, $X_{\ominus A} \Vdash\!\not= Y$ and thus the set $X_{\ominus A}, \neg Y$ is consistent, that is, $Z_{\ominus A}$ is consistent.

Now assume that $A \in \neg Y$. Hence $A = \neg B$ for some $B \in Y$. By the clause 3 of Definition 4, $X \Vdash\!\not= Y_{\ominus B}$. Thus the set $X, \neg(Y_{\ominus B})$ is consistent. Yet, $X, \neg(Y_{\ominus B}) = Z_{\ominus A}$. Hence the set $Z_{\ominus A}$ is consistent.

By Corollary 8, $X, \neg Y$ is thus a MI-set.

($\Leftarrow$) Assume that $X \cap \neg Y = \emptyset$ and $X, \neg Y$ is a MI-set. From the latter it follows that $X \Vdash\!= Y$.

Again, let $Z = X, \neg Y$.

Suppose that $X_{\ominus A} \Vdash\!= Y$ for some $A \in X$. Then the set $X_{\ominus A}, \neg Y$ is inconsistent. Yet, since $X \cap \neg Y = \emptyset$, the set $X_{\ominus A}, \neg Y$ is a proper subset of Z. Thus Z is not a MI-set. A contradiction.

Now suppose that $X \Vdash\!= Y_{\ominus B}$ for some $B \in Y$. Let us designate $Y_{\ominus B}$ by Y^*. As $X \Vdash\!= Y^*$ holds, the set $X, \neg Y^*$ is inconsistent. But $X \cap \neg Y = \emptyset$, so $\neg B$ does not belong to X. Hence the set $X, \neg Y^*$ is a proper subset of Z. Thus Z is not a MI-set. A contradiction again.

Therefore $X \Vdash\!\prec Y$. □

Theorem 5 yields:

Corollary 9.

1. *$X \Vdash\!\prec \emptyset$ iff X is a* MI*-set.*
2. *$\emptyset \Vdash\!\prec Y$ iff $\neg Y$ is a* MI*-set.*

Remark 4. As the second part of the proof of Theorem 5 shows, one can get $X \Vdash\!\prec Y$ from the fact that $X, \neg Y$ is a MI-set *on the condition* that $X \cap \neg Y = \emptyset$ holds. This condition is a necessary one. For example, let $X = \{p \vee q, \neg p, \neg q\}$ and $Y = \{p, q\}$. Then $\neg Y = \{\neg p, \neg q\}$ and hence $X, \neg Y = X$. As X is a MI-set, so is $X, \neg Y$. However, $X \Vdash\!\not\prec Y$, since $\{p \vee q\} \Vdash\!= \{p, q\}$. On the other hand, $X \cap \neg Y = \{\neg p, \neg q\} \neq \emptyset$.

There exist MI-sets which do not contain wffs beginning with negation, i.e. wffs of the form $\neg B$. Here are simple examples:

$$\{p \to q, p \wedge \neg q\}$$
$$\{p, p \to q, p \to \neg q\}$$

It may seem that such MI are "useless" in showing that strong mc-entailment holds. But this is wrong. The corollary below explains why.

Corollary 10. *If X, Y is a MI-set and $X \cap Y = \emptyset$, then $X \Vert\!\prec \neg Y$.*

Proof. Clearly, if X, Y is a MI-set, then $X, \neg(\neg Y)$ is a MI-set. Suppose that $X \cap \neg(\neg Y) \neq \emptyset$. So there exists $A \in X$ such that $A = \neg\neg B$ for some $B \in Y$, and $A \in \neg(\neg Y)$. As X, Y is a MI-set and $X \cap Y = \emptyset$, we have $B \notin X$ and thus the set $X, Y_{\ominus B}$ is consistent. Hence $X, (\neg(\neg Y))_{\ominus\neg\neg B}$ is a consistent set as well. But $X, (\neg(\neg Y))_{\ominus\neg\neg B} = X, \neg(\neg Y)$, since $A = \neg\neg B$ and $A \in X$. It follows that $X, \neg(\neg Y)$ is not a MI-set. We arrive at a contradiction. Thus $X \cap \neg(\neg Y) = \emptyset$. As $X, \neg(\neg Y)$ is a MI-set, by Theorem 5 we get $X \Vert\!\prec \neg Y$. □

3.2.4 Independence and Deduction

Observe that if X strongly mc-entails Y, then neither X nor Y contains syntactically distinct wffs which are logically equivalent, i.e. entail each other. The reason is that a MI-set never includes logically equivalent wffs. We can also prove more:

Theorem 6 (Independence)**.** *Let $X \Vert\!\prec Y$, and let A, B be syntactically distinct wffs.*

1. *If $A, B \in X$ and $Y \neq \emptyset$, then $A \not\models B$ and $A \not\models \neg B$.*
2. *If $A, B \in Y$, then $A \not\models B$, and $\neg A \not\models B$ provided that $\{A, B\} \neq Y$.*

Proof. If $X \Vert\!\prec Y$, then, by Theorem 5, $X, \neg Y$ is a MI-set and $X \cap \neg Y = \emptyset$.

Let $A, B \in X$. Thus the set $X_{\ominus B}, \neg Y$ is consistent and, due to the fact that $X, \neg Y$ is inconsistent, $X_{\ominus B}, \neg Y \models \neg B$. But $A \in X_{\ominus B}$. Therefore $X_{\ominus B}, \neg Y \models A$. Hence $A \not\models B$.

As $B \in X$, we have $X \models B$. Suppose that $A \models \neg B$. Since $A \in X$, it follows that $X \models \neg B$. Thus X is an inconsistent set and, as $Y \neq \emptyset$, we get $X \Vert\!\not\prec Y$.

Let $A, B \in Y$. It follows that $\neg A, \neg B \in \neg Y$. By Theorem 5, $X, \neg Y$ is a MI-set and hence an inconsistent set. Thus $X, \neg(Y_{\ominus A}) \models A$. However, the set $X, \neg(Y_{\ominus A})$, as a proper subset of the MI-set in question, is consistent. On the other hand, '$\neg B$' $\in \neg(Y_{\ominus A})$. It follows that $X, \neg(Y_{\ominus A}) \models \neg B$. Therefore $A \not\models B$.

Assume that $\{A, B\} \neq Y$. Suppose that $\neg A \models B$. It follows that $\emptyset \models \neg A \to B$ and hence $\emptyset \Vert\!\models \{A, B\}$. As $\{A, B\} \neq Y$, we get $X \Vert\!\not\prec Y$. □

Notation. For conciseness, let us introduce the following notational convention:

$$\lceil A \to W \rceil =_{df} \quad \begin{cases} \{\neg A\} \ \ if \ \ W = \emptyset, \\ \{A \to B : B \in W\} \ \ if \ \ W \neq \emptyset. \end{cases}$$

One can easily show that the following holds:

Corollary 11. *$Z, A \Vert\!\models W$ iff $Z \Vert\!\models \lceil A \to W \rceil$.*

As a consequence we get:

Theorem 7 (Deduction for strong mc-entailment)**.** *Let $A \notin X$. If $X, A \Vert\!\prec Y$, then $X \Vert\!\prec \lceil A \to Y \rceil$.*

Proof. Assume that $X, A \Vert\!\prec Y$. If $X, A \Vert\!\models Y$, then, by Corollary 11, $X \Vert\!\models \lceil A \to Y \rceil$. Let $B \in X$. Since, by assumption, $A \notin X$, it follows that $A \neq B$. Thus $X_{\ominus B}, A$ is a proper subset of X, A. As $X, A \Vert\!\prec Y$ holds, we have $X_{\ominus B}, A \Vert\!\not\models Y$. Hence, by Corollary 11 again, $X_{\ominus B} \Vert\!\not\models \lceil A \to Y \rceil$. Let $C \in Y$. Thus $X, A \Vert\!\not\models Y_{\ominus C}$. Therefore, by Corollary 11, $X \Vert\!\not\models \lceil A \to Y_{\ominus C} \rceil$. Hence $X \Vert\!\prec \lceil A \to Y \rceil$. □

Note that the converse of Theorem 7 is not true. For example, $\emptyset \Vert\!\prec \{p \to q, p \to \neg q\}$ holds, but $\{p\} \Vert\!\prec \{q, \neg q\}$ is not the case. However, the following is true:

Corollary 12. *If $X \Vert\!\prec \lceil A \to Y \rceil$ and $X \not\models \neg A$ as well as $X \Vert\!\not\models Y$, then $X, A \Vert\!\prec Y$.*

Proof. Suppose that $Y = \emptyset$. Hence $X \Vert\!\prec \{\neg A\}$. Thus $X \models \neg A$. But, by assumption, $X \not\models \neg A$. So $Y \neq \emptyset$.

If $X \Vert\!\prec \lceil A \to Y \rceil$, then, by Definition 4 and Corollary 11, $X, A \Vert\!\models Y$ and $X_{\ominus B} \cup \{A\} \Vert\!\not\models Y$ for any $B \in X$. By assumption, $X \Vert\!\not\models Y$. It follows that for every $C \in X, A$ we have $X, A \setminus \{C\} \Vert\!\not\models Y$. Now suppose that $X, A \Vert\!\models Y_{\ominus D}$ is the case for some $D \in Y$. There are two possibilities: (a) $Y_{\ominus D} = \emptyset$ and (b) $Y_{\ominus D} \neq \emptyset$. Assume that (a) holds. It follows that the set X, A is inconsistent. But, by assumption, $X \not\models \neg A$ and hence the set X, A is consistent. So (a) does not hold. It follows that Y is not a singleton set. Assume that (b) is the case. Therefore, by Corollary 11, $X \Vert\!\models \lceil A \to Y_{\ominus D} \rceil$. As $\lceil A \to Y_{\ominus D} \rceil$ is a proper subset of $\lceil A \to Y \rceil$, it follows that $X \Vert\!\not\prec \lceil A \to Y \rceil$. So we arrive at a contradiction again. Hence $X, A \Vert\!\not\models Y_{\ominus D}$ for every $D \in Y$. As all the clauses of Definition 4 are fulfilled w.r.t. X, A and Y, we conclude that $X, A \Vert\!\prec Y$ holds. □

As the proof of Corollary 12 shows, the assumption "$X \not\models \neg A$" is dispensable when Y is neither a singleton set nor the empty set.

Corollary 13. *Let Y be a finite and at least two-element set of wffs. If $X \Vdash\!\prec \lceil A \to Y \rceil$ and $X \not\Vdash\!= Y$, then $X, A \Vdash\!\prec Y$.*

Finally, observe that $\Vdash\!\prec$ is not closed under uniform substitution. A simple example illustrates this. Clearly, $\{p\} \Vdash\!\prec \{p\}$ is the case. But $\{p \wedge \neg p\} \Vdash\!\prec \{p \wedge \neg p\}$ does not hold (cf. Corollary 6). Needless to say, $p \wedge \neg p$ results from p by substitution.

4 Strong Single-Conclusion Entailment

4.1 Definition and the Adequacy Issue

Sc-entailment traditionally construed can be identified with mc-entailment of a singleton set. Similarly, it seems natural to define strong sc-entailment as strong mc-entailment of a singleton set.

We use $\models\!\prec$ as the symbol for strong sc-entailment.

Definition 6 (Strong sc-entailment)**.** $X \models\!\prec B$ *iff* $X \Vdash\!\prec \{B\}$.

For brevity, we will write $A \models\!\prec B$ instead of $\{A\} \models\!\prec B$.

As an immediate consequence of Definition 6 and Theorem 5 one gets:

Theorem 8. $X \models\!\prec B$ *iff* '$\neg B$' $\notin X$ *and* $X, \neg B$ *is a* MI*-set.*

Note that the transition from right to left requires '$\neg B$' $\notin X$ to hold. For example, although

$$\{p, p \to \neg q, \neg\neg q\} \cup \{\neg\neg q\} \tag{28}$$

is a MI-set, $\{p, p \to \neg q, \neg\neg q\} \models\!\prec \neg q$ does not hold, since '$\neg\neg q$'$\in \{p, p \to \neg q, \neg\neg q\}$.

The following is true:

Corollary 14. $X \models\!\prec B$ *iff*

1. *$X \models B$ and*
2. *for each proper subset Z of X: $Z \not\models B$, and*
3. *X is consistent.*

Proof. Clearly, $X \models B$ holds iff $X \Vdash\!= \{B\}$ is the case.

Clause 2 holds due to Corollary 1. On the other hand, clause 2 yields that there is no $A \in X$ such that $X_{\ominus A} \Vdash\!= \{B\}$.

Since $\{B\} \setminus \{B\} = \emptyset$, clause 3 of Definition 4 and clause 3 of the above corollary are equivalent for $Y = \{B\}$. $\square$

Strong sc-entailment is not monotone. As a matter of fact, it is “antimonotone” in a sense explained by:

Corollary 15. *If $X \mathrel{|\!\!<} B$ and $X \subseteq Y$, where $Y \neq X$, then $Y \mathrel{|\!\!\not<} B$.*

Proof. By Definition 6 and Corollary 2. □

As we pointed out in section 1.1, the monotonicity of entailment contravenes, in a sense, the semantic entrenchment idea, since it allows semantically irrelevant wffs to occur among premises. In the case of strong sc-entailment, however, the difficulty is solved in a radical way: a strongly sc-entailing set is “minimal” with regard to the transmission of truth and, since no proper superset of a set X that strongly sc-entails a wff B strongly sc-entails B as well, adding an “irrelevant” wff to X results in the lack of strong sc-entailment of B from X enriched in this way.

By the clause 2 of Corollary 14, each proper subset of a strongly sc-entailing set is consistent. Strong sc- and mc-entailment do not differ in this respect. As we have seen, however, there exist strongly mc-entailing sets which are inconsistent (each of them strongly mc-entails only the empty set, however). According to the clause 3 of Corollary 14, this never happens in the case of strong sc-entailment. Anyway, strong sc-entailment is free of the drawback (I) pointed out in section 1.1. Let us add: free, again, in a radical way, since inconsistent sets do not strongly sc-entail any wffs. As an immediate consequence of Corollary 6 one gets:

Corollary 16. *No wff is strongly sc-entailed by an inconsistent set of wffs.*

Thus no inconsistent wff belongs to a sc-entailing set, and a singleton set which comprises an inconsistent wff does not strongly sc-entail any wff. In particular, neither $A \wedge \neg A \mathrel{|\!\prec} A$ nor $\{A, \neg A\} \mathrel{|\!\prec} A$ holds, regardless of what A is. Similarly, there is no B such that $A \wedge \neg A \mathrel{|\!\!<} B$ or $\{A, \neg A\} \mathrel{|\!\!<} B$.

Observe that the following holds as well:

Corollary 17. *There is no set of wffs that strongly sc-entails an inconsistent wff.*

Proof. By Definition 6 and Theorem 2. □

Thus inconsistencies are outside the realm of strong sc-entailment: no inconsistent set belongs to the domain of $\mathrel{|\!\!<}$ and no inconsistent wff belongs to the range of the relation. No doubt, a paraconsistent logician will dislike strong sc-entailment.

The case of validities is slightly more complicated. By Theorem 2 we get:

Corollary 18. *If $X \mathrel{|\!\!<} B$, then no wff in X is valid.*

Corollary 19. *If B is valid and $X \mathrel{|\!\!<} B$, then $X = \emptyset$.*

One can prove that valid wffs are exactly these wffs which are strongly sc-entailed only by the empty set.

Corollary 20. *A wff B is valid iff $\emptyset \mathrel{|\!\!\prec} B$ and $X \mathrel{|\!\!\not\prec} B$ for any $X \neq \emptyset$.*

Proof. Let B be a valid wff. Thus $\{\neg B\}$ is a MI-set, and hence, by Corollary 9, $\emptyset \mathrel{|\!\!\prec} B$. Thus $X \mathrel{|\!\!\not\prec} B$ for any $X \neq \emptyset$. On the other hand, if $\emptyset \mathrel{|\!\!\prec} B$, then $\emptyset \models B$ and hence B is valid. □

As for valid wffs, Corollary 19 yields that the difference between strong sc-entailment and sc-entailment simpliciter lies in the fact that valid wffs are strongly sc-entailed *only* by the empty set. Thus, in particular, valid wffs are not strongly sc-entailed by sets of valid wffs. Moreover, a valid wff is not sc-entailed by any set of wffs to which a valid wff belongs to.

4.2 Some Properties of Strong Sc-entailment

Since strong sc-entailment is defined in terms of strong mc-entailment, one can easily derive the following corollaries from the corresponding results presented in sections 3.2.1, 3.2.2, and 3.2.4.

Corollary 21. *Let A, B be logically equivalent wffs.*

1. *If $A \in X$ and $X \mathrel{|\!\!\prec} C$, then $X_{\ominus A} \cup \{B\} \mathrel{|\!\!\prec} C$.*
2. *If $X \mathrel{|\!\!\prec} A$, then $X \mathrel{|\!\!\prec} B$.*

Corollary 22 (Contingency for $\mathrel{|\!\!\prec}$)**.** *If $X \mathrel{|\!\!\prec} B$ and $X \neq \emptyset$, then each wff in $X \cup \{B\}$ is contingent.*

Corollary 23 (Strict finiteness of $\mathrel{|\!\!\prec}$)**.** *If $X \mathrel{|\!\!\prec} B$, then X is a finite set.*

Corollary 24 (Variable sharing for $\mathrel{|\!\!\prec}$)**.** *If $X \mathrel{|\!\!\prec} B$ and $X \neq \emptyset$, then $\mathtt{Var}(X) \cap \mathtt{Var}(B) \neq \emptyset$.*

Corollary 25 (Independence for $\mathrel{|\!\!\prec}$)**.** *Let $X \mathrel{|\!\!\prec} B$. If A, C are syntactically distinct wffs that belong to X, then $A \not\models C$ and $A \not\models \neg C$.*

Corollary 26 (Deduction for strong sc-entailment)**.** *Let $A \notin X$. If $X, A \mathrel{|\!\!\prec} B$, then $X \mathrel{|\!\!\prec} A \to B$.*

The converse of Corollary 26 is not true. For instance, $\emptyset \mathrel{|\!\!\prec} p \wedge \neg p \to q$ holds, but $p \wedge \neg p \mathrel{|\!\!\prec} q$ does not hold. Yet, there are cases in which $X \mathrel{|\!\!\prec} A \to B$ yields $X, A \mathrel{|\!\!\prec} B$. Corollary 12 implies:

Corollary 27. *If* $X \mid\!\!< A \to B$*, and* $X \not\models \neg A$ *as well as* $X \not\models B$*, then* $X, A \mid\!\!< B$*.*

Thus we get:

Corollary 28. *Let* $A \notin X$*, and* $X \not\models \neg A$ *as well as* $X \not\models B$*. Then* $X, A \mid\!\!< B$ *iff* $X \mid\!\!< A \to B$*.*

Proof. By corollaries 26 and 27. □

Strong sc-entailment is, in a sense, closed under detachment.

Corollary 29 (Detachment for $\mid\!\!<$)**.** *If* $X \mid\!\!< A \to B$ *and* $X \mid\!\!< A$*, then* $X \mid\!\!< B$*.*

Proof. Either $X \mid\!\!< A \to B$ or $X \mid\!\!< A$ warrants the consistency of X, and together they yield that $X \models B$ holds.

Assume that $X \neq \emptyset$. Let C be an arbitrary but fixed element of X. From $X \mid\!\!< A \to B$ we get $X_{\ominus C} \not\models A \to B$. It follows that $X_{\ominus C} \not\models B$. Thus $X \mid\!\!< B$.

Now assume that $X = \emptyset$. In this case B is a valid wff. Therefore $\emptyset \mid\!\!< B$ due to Corollary 20, that is, $X \mid\!\!< B$. □

Observe that one can also prove that $X \mid\!\!< A \to B$ and $X \models A$ yield $X \mid\!\!< B$.

4.2.1 Strong Sc-entailment from Singleton Sets

Strong sc-entailment from single wffs (more precisely, from singleton sets of wffs) has some properties which strong sc-entailment from non-singleton sets lack.

Corollary 30. *The following are equivalent:*

1. $A \mid\!\!< B$*,*
2. $A \models B$ *and* A, B *are contingent wffs.*

Proof. The implication from (*1*) to (*2*) is due to Definition 6 and Corollary 22. As for the passage from (*2*) to (*1*), it suffices to observe that the contingency of A warrants the consistency of $\{A\}$, while the contingency of B guarantees that $\emptyset \parallel\!\!\models \{B\}$ does not hold. □

One cannot generalize Corollary 30 to non-singleton sets. The contingency of all the wffs belonging to a (non-empty) non-singleton set of wffs X warrants neither the consistency of X itself nor the lack of entailment of B from proper subset(s) of X.

Coming back to sc-entailment from single wffs. The lack of strong sc-entailment in the presence of standard sc-entailment tells us more about the wffs involved than Corollary 30 does.

Corollary 31. *If $A \models B$, but $A \mid\!\nprec B$, then A is inconsistent or B is valid.*

Proof. If $A \models B$ and $A \mid\!\nprec B$, then $\{A\}$ is an inconsistent set or $\emptyset \models B$. So A is inconsistent or B is valid. □

When X is a non-empty set having more that one element, the lack of $X \mid\!\prec B$ in the presence of $X \models B$ implies that X is inconsistent or B is entailed by some proper subset of X.

Finally, let us notice the following:

Corollary 32. *If $A \mid\!\prec B$ and $B \mid\!\prec C$, then $A \mid\!\prec C$.*

Proof. Certainly, $A \models B$ and $B \models C$ yields $A \models C$. By Corollary 30, $A \mid\!\prec B$ warrants the contingency of A, while $B \mid\!\prec C$ yields the contingency of C. So $A \mid\!\prec C$ due to Corollary 30. □

Observe that when X has more than one element, the passage from $X \mid\!\prec B$ and $B \mid\!\prec C$ to $X \mid\!\prec C$ requires an additional condition to be met, namely it must be ensured that for each $D \in X$, the set $X_{\ominus D}$ does not entail C.

4.2.2 Mutuality

As for CPL, mc-entailment of a non-empty finite set reduces to sc-entailment of a disjunction of all the elements of the set, i.e. if Y is a finite set and $Y \neq \emptyset$, then $X \Vert\!\models Y$ iff $X \models \bigvee Y$. But strong mc-entailment and strong sc-entailment are not linked in this way. For instance, we have:

$$p \mid\!\prec p \vee q \tag{29}$$

but we *do not* have:[11]

$$p \Vert\!\prec \{p, q\} \tag{30}$$

Strong mc- and sc-entailments are mutually linked in a quite different way, as the following theorem shows.

Theorem 9 (Mutuality)**.**

1. *If $X \Vert\!\prec Y, B$, where $B \notin Y$, then $X, \neg Y \mid\!\prec B$.*

2. *If $X, \neg Y \mid\!\prec B$ and $X \cap \neg Y = \emptyset$, then $X \Vert\!\prec Y, B$.*

[11] (30) does not hold because $\{p\} \Vert\!\models (\{p, q\} \setminus \{q\})$.

Proof. If $X \Vdash\!\!\prec Y, B$, then, by Theorem 5, $X \cup (\neg Y \cup \{\neg B\})$ is a MI-set and $X \cap (\neg Y \cup \{\neg B\}) = \emptyset$. It follows that $(X \cup \neg Y) \cup \{\neg B\}$ is a MI-set and '$\neg B$' $\notin X$. By assumption, $B \notin Y$. So '$\neg B$' $\notin \neg Y$. Hence '$\neg B$' $\notin X, \neg Y$. Thus $X, \neg Y \models\!\!\prec B$ by Theorem 8.

If $X, \neg Y \models\!\!\prec B$, then, by Theorem 8, $(X \cup \neg Y) \cup \{\neg B\}$ is a MI-set and '$\neg B$' $\notin X \cup \neg Y$. Suppose that $X \cap (\neg Y \cup \{\neg B\}) \neq \emptyset$. As '$\neg B$' $\notin X \cup \neg Y$, it follows that $(X \cap \neg Y) \neq \emptyset$. On the other hand, by assumption $(X \cap \neg Y) = \emptyset$. Therefore $X \Vdash\!\!\prec Y, B$ due to Theorem 5. □

4.2.3 Conjunction vs. Set of Conjuncts

As for the standard sc-entailment based on Classical Logic, there is no scope difference between being entailed by a finite set of wffs and being entailed by a conjunction of all the wffs of this set. Although conjunction, $\wedge$, is semantically construed here in the classical manner (cf. Section 2), it is worth to note that strong sc-entailment from a conjunction of wffs and strong sc-entailment from a set of all its conjuncts only overlap, but not coincide. Clearly, the following is true:

Corollary 33. *Let $X \neq \emptyset$. If $X \models\!\!\prec B$, then $\bigwedge X \models\!\!\prec B$.*

For example, $\{p, q\} \models\!\!\prec p \wedge q$ is the case and thus $p \wedge q \models\!\!\prec p \wedge q$ holds as well. Yet, the converse of Corollary 33 is not true. For instance, $p \wedge q \models\!\!\prec p$ holds, while $\{p, q\} \models\!\!\prec p$ does not hold.[12] At first sight this looks untenable. However, the phenomenon can be explained as follows. Information carried by $\bigwedge X \models\!\!\prec B$ and $X \models\!\!\prec B$ differ when X is not a singleton set. In both cases transmission of truth as well as consistency of the set X are ensured. The claim of $\bigwedge X \models\!\!\prec B$ is: although B need not be true, the (hypothetical) truth of all the wffs in X is sufficient for B be true. Note that $\bigwedge X \models\!\!\prec B$ does not exclude that the transmission of truth effect takes place w.r.t. some proper subset or some proper superset of X. (As for $p \wedge q \models\!\!\prec p$, there is a proper subset of $\{p, q\}$, namely $\{p\}$, which ensures the transmission.) The claim of $X \models\!\!\prec B$ is stronger: this is just the (hypothetical) truth of all the wffs in X that warrants the (hypothetical) truth of B. "Just" means here: "one needs neither more nor less than the truth of *all* the wffs in X for B be true."

Observe that one can pass from $\bigwedge X \models\!\!\prec B$ to $X \models\!\!\prec B$ on the condition:

($\heartsuit$) *for each $A \in X : \bigwedge(X_{\ominus A}) \not\models B$*

which, however, does not hold universally.

[12] By the way, these examples provide a nice illustration of the lack of transitivity of strong sc-entailment.

Remark 5. Although strong sc-entailment is "antimonotone" (cf. Corollary 15), the following fact is worth some attention:

Corollary 34. *Let $X \neq \emptyset$. If $X \models\!\!\!< B$ and Y is a consistent proper superset of X, then $\bigwedge Y \models\!\!\!< B$.*

Proof. By Corollary 19, if $X \neq \emptyset$ and $X \models\!\!\!< B$, then B is not valid. So $\emptyset \not\models B$. On the other hand, $\emptyset$ is the only proper subset of $\{\bigwedge Y\}$. Clearly, if $X \models B$, then $Y \models B$ and hence $\{\bigwedge Y\} \models B$. If Y is consistent, so is $\{\bigwedge Y\}$. Therefore $\bigwedge Y \models\!\!\!< B$. □

Thus a wff strongly sc-entailed by a non-empty set of wffs X is also strongly sc-entailed by (the singleton set comprising) a conjunction of all the wffs of a consistent extension Y of X. Note, however, that, according to what has been said above, $\bigwedge Y \models\!\!\!< B$ carries less information than $X \models\!\!\!< B$. Moreover, $X \models\!\!\!< B$ suppresses $Y \models\!\!\!< B$.

5 Some Comparisons

5.1 Strong vs. Classical

The basic properties of strong entailments differ from those of their classical counterparts. However, one can show that whatever is reachable by classical entailments from consistent sets of premises, is also attainable by strong entailments from some finite subsets of these sets. To put it briefly: no classical consequence of a consistent set is lost.

Notice that it holds that (we present a proof of this well-known fact only to keep this paper self-contained):

Lemma 1. *Each inconsistent set of wffs has a subset being a* MI*-set.*

Proof. Let X be an inconsistent set of wffs. By compactness of CPL, X has an inconsistent finite subset, say, X'. Clearly, $X' \neq \emptyset$. Consider the family of all inconsistent subsets of X'. Let us designate it by Ψ. Since X' is inconsistent, $\Psi \neq \emptyset$. As X' is non-empty and finite, there is a natural number, say, k, where $k \geqslant 1$, such that no set in Ψ has less than k elements. Let Y be an element of Ψ which comprises exactly k wffs. Obviously, no proper subset of Y belongs to Ψ. Therefore each proper subset of Y is consistent. It follows that Y is a MI-set included in X. □

Let us now prove:

Theorem 10 (Simulation of $\Vert\!\models$)**.** *If $X \Vert\!\models Y$ and X is consistent, then there exist a finite subset X_1 of X and a finite non-empty subset Y_1 of Y such that $X_1 \Vert\!\models\!\!\!< Y_1$.*

Proof. If $X \Vert\!\!\models Y$, then the set $X, \neg Y$ is inconsistent and thus, by Lemma 1, has subset(s) being MI-sets. Let Z be a MI-set such that $Z \subseteq X, \neg Y$. Since, by assumption, X is consistent, $Z \nsubseteq X$. We put:

$$X_1 =_{df} X \cap Z$$
$$W =_{df} Z \setminus X_1$$

Clearly, $W \subseteq \neg Y$. Moreover, $W \neq \emptyset$, and $Z = X_1, W$ as well as $X_1 \cap W = \emptyset$. Consider the set Y_1 defined by:

$$Y_1 =_{df} \{C : \neg C \in W\}.$$

We have $W = \neg Y_1$ and hence $Z = X_1, \neg Y_1$. It follows that $Y_1 \subseteq Y$ and $X_1 \cap \neg Y_1 = \emptyset$. Since Z is a MI-set and $Z = X_1, \neg Y_1$ as well as $X_1 \cap \neg Y_1 = \emptyset$, by Theorem 5 we conclude that $X_1 \Vert\!\!\prec Y_1$ holds. As each MI-set is finite, X_1 and Y_1 are finite subsets of X and Y, respectively. Finally, $Y_1 \neq \emptyset$ since $W \neq \emptyset$. □

As a consequence of Definition 6 and Theorem 10 we get:

Theorem 11 (Simulation of $\models$). *If $X \models B$ and X is consistent, then there exists a finite subset Z of X such that $Z \mid\!\!\prec B$.*

Proof. Recall that $X \models B$ iff $X \Vert\!\!\models \{B\}$, and $Z \mid\!\!\prec B$ iff $Z \Vert\!\!\prec \{B\}$. Since we have already proven Theorem 10, it suffices to observe that the only non-empty subset of the singleton set $\{B\}$ is $\{B\}$ itself. □

The intuitive content of Theorem 11 is this: CPL sc-entailment from a given, finite or infinite, consistent set of wffs boils down to strong sc-entailment from a finite subset of the set. Theorem 10 presents an analogous result for mc-entailment.

Remark 6. Let X and Y be different, yet logically equivalent consistent sets of wffs. The set of wffs classically sc-entailed by X coincides with the set of wffs classically entailed by Y. However, this need not be the case for strong sc-entailment. Yet, Theorem 11 yields that the set of wffs attainable by strong sc-entailment from some finite subset of X equals the set of wffs which are obtainable by strong sc-entailment from some finite subset of Y, and equals the set comprising all the wffs classically sc-entailed by X or by Y. Needless to say, the respective subsets of X and of Y may differ.

5.2 Strong vs. Relevant

As it is well-known, when the sum of two consistent sets of CPL-wffs, X and Y, is inconsistent, then $\mathtt{Var}(X) \cap \mathtt{Var}(Y) \neq \emptyset$ (cf. e.g. [6], p. 375). It follows that classical sc-entailment from consistent sets of premises to conclusions which are not valid wffs exhibits the variable sharing property. It is worth to note that the same holds true for strong sc-entailment. Theorem 11 together with corollaries 24 and 20 almost immediately yield:

Corollary 35. *Let X be a non-empty, consistent set of wffs. If $X \models B$ and B is not a valid wff, then there exists a finite, non-empty subset Z of X such that $Z \models\!\!\!< B$ and $\mathtt{Var}(Z) \cap \mathtt{Var}(B) \neq \emptyset$.*

Variable sharing is often regarded as an indicator (or even a precondition) of relevance in the context of semantic consequence. As such, it is usually invoked in relevant logics. So the question arises: what is the relation between strong sc-entailment and accounts of entailment proposed in relevant logics? Since there exist many systems of relevance logic, an exhaustive answer would have required a separate paper. For the reasons of space, let me restrict to a few remarks only.

As for CPL, valid wffs falling under the schema:

$$A \to B \tag{31}$$

license sc-entailment of B from A. For the lack of a better idea, let us call them *classical implicational laws* or briefly CIL's.[13]

Recall that although all classically valid wffs are strongly sc-entailed by the empty set, the transition from $\emptyset \models\!\!\!< A \to B$ to $A \models\!\!\!< B$ is not always legitimate (cf. Corollary 28). Corollary 30 yields, in turn, that a CIL does not license strong sc-entailment just in case its antecedent or consequent is not contingent.

The first observation is: there exist CIL's which are both rejected in some relevant logics[14] and do not license strong sc-entailment. Examples are shown in Table 1.

Second, there exist CIL's which are rejected in some relevant logic(s), but license strong sc-entailment. Examples are given in Table 2.

Third, it happens that a CIL which is accepted in a relevant logic does not license strong sc-entailment. The "mingle" formulas, i.e. wffs of the form $A \to (A \to A)$, provide simple examples here.

[13]One should not confuse CIL's with laws of the implicational fragment of CPL. Both A and B may involve any connective, implication included. What is important is that implication is the main connective of (the wff which expresses) a CIL.

[14]That is, at least one of well-known relevant logics rejects the corresponding law; there is no space for details. For relevant logics see, e. g., [13] and [18].

$p \wedge \neg p \to q$	$p \wedge \neg p \not\mathrel{|\!\!<} q$
$\neg(p \to p) \to q$	$\neg(p \to p) \not\mathrel{|\!\!<} q$
$p \to (q \to q)$	$p \not\mathrel{|\!\!<} q \to q$
$p \to (p \to p)$	$p \not\mathrel{|\!\!<} p \to p$
$p \to p \vee \neg p$	$p \not\mathrel{|\!\!<} p \vee \neg p$
$p \to q \vee \neg q$	$p \not\mathrel{|\!\!<} q \vee \neg q$
$(p \to p) \to (q \to q)$	$p \to p \not\mathrel{|\!\!<} q \to q$
$(p \to q) \to (p \to p)$	$p \to q \not\mathrel{|\!\!<} p \to p$

Table 1: *Examples of* CIL*'s rejected in some relevant logics (left column) which do not license strong sc-entailment (as depicted in the right column).*

$p \to (q \to p)$	$p \mathrel{|\!\!<} (q \to p)$
$p \to (\neg p \to q)$	$p \mathrel{|\!\!<} (\neg p \to q)$
$p \to ((p \to q) \to q)$	$p \mathrel{|\!\!<} (p \to q) \to q$
$((p \to q) \to p) \to p$	$(p \to q) \to p \mathrel{|\!\!<} p$
$(p \to (q \to r)) \to (q \to (p \to r))$	$p \to (q \to r) \mathrel{|\!\!<} q \to (p \to r)$
$p \wedge q \to (p \to q) \wedge (q \to p)$	$p \wedge q \mathrel{|\!\!<} (p \to q) \wedge (q \to p)$
$p \wedge (\neg p \vee q) \to q$	$p \wedge (\neg p \vee q) \mathrel{|\!\!<} q$

Table 2: *Examples of* CIL*'s rejected in some relevant logics (left column) which, however, license strong sc-entailment (as depicted in the right column).*

5.3 Strong vs. Connexive

Connexive logics are usually characterized as systems validating the following theses:[15]

$$\neg(\neg A \to A) \tag{32}$$

$$\neg(A \to \neg A) \tag{33}$$

$$(A \to B) \to \neg(A \to \neg B) \tag{34}$$

$$(A \to \neg B) \to \neg(A \to B) \tag{35}$$

As strong sc-entailment from the empty set is restricted to classically valid wffs only (cf. Corollary 20) and some wffs of the forms (32 – 35) are not classically valid, it is not the case that all the wffs falling under the schemata (32) – (35) are strongly

[15]For connexive logics see, e.g. [14]. Theses (32) and (33) are attributed to Aristotle, while theses (34) and (35) are ascribed to Boethius.

sc-entailed by the empty set.[16] It is worth to note, however, that the following are true:

Corollary 36. *For any wff A:*

1. $\neg A \not\models\!\!< A$,
2. $A \not\models\!\!< \neg A$.

Proof. Suppose that $\neg A \models\!\!< A$ for some wff A. Thus $\neg A \models A$. On the other hand, by Theorem 1 it follows that both $\neg A$ and A are contingent wffs. Hence $\neg A \not\models A$.
We reason analogously in the case of (*2*). □

Thus a negation of a wff never strongly sc-entails the wff itself, and a wff never strongly sc-entails its negation. Corollary 36 seems to express an idea akin to that which lies behind having (32) and (33) as theses.

Corollary 37. *For any wffs A, B:*

1. *if* $A \models\!\!< B$, *then* $A \not\models\!\!< \neg B$,
2. *if* $A \models\!\!< \neg B$, *then* $A \not\models\!\!< B$,

Proof. Assume that $A \models\!\!< B$. It follows that A is a consistent wff and $A \models B$. Therefore there exists a valuation v such that $v(A) = \mathbf{1}$, $v(B) = \mathbf{1}$ and hence $v(\neg B) = \mathbf{0}$. Thus $A \not\models \neg B$. It follows that $A \not\models\!\!< \neg B$.
We reason similarly in the case of (*2*). □

A due comment on Corollary 37 is analogous to that on Corollary 36.

6 Towards a Proof-theoretic Account of Strong Entailments

As Theorem 5 shows, a problem of the form:

(**P**) *Does X strongly mc-entail Y?*

splits into two sub-problems:

(P$_1$) *Is it the case that $X \cap \neg Y = \emptyset$?*

[16]However, some of them are classically valid and thus *are* strongly sc-entailed by the empty set. For instance, we have $\emptyset \models\!\!< \neg(\neg(p \wedge \neg p) \to p \wedge \neg p)$, $\emptyset \models\!\!< \neg((p \to p) \to \neg(p \to p))$, or $\emptyset \models\!\!< (p \vee \neg p \to p \wedge \neg p) \to \neg(p \vee \neg p \to \neg(p \wedge \neg p))$.

($\mathbf{P}_2$) *Is* $X, \neg Y$ *a* MI-*set?*

Similarly, due to Theorem 8, a problem of the form:

(**P**') *Does* X *strongly sc-entail* B*?*

splits into:

(P'$_1$) *Is it the case that '*$\neg B$*'* $\notin X$*?*

(P'$_2$) *Is* $X, \neg B$ *a* MI-*set?*

$\mathbf{P}_1$ and $\mathbf{P}'_1$ are purely syntactic issues. But either $\mathbf{P}_2$ or $\mathbf{P}'_2$ is a problem that pertains to a semantic property. In order to solve it syntactically one needs a proof-theoretic account of MI-sets.

6.1 The Calculus $\mathsf{MI}^{\mathsf{CPL}}$

In this section we present a calculus, labelled $\mathsf{MI}^{\mathsf{CPL}}$, in which provable sequents of a strictly defined form correspond to MI-sets.

Rules of the calculus operate on sequences of sequents of a specific kind. Since a sequence of sequents is customarily called a hypersequent, $\mathsf{MI}^{\mathsf{CPL}}$ may be called a calculus of hypersequents. But speaking about hypersequent calculi usually brings into mind Avron's seminal works.[17] However, the format of $\mathsf{MI}^{\mathsf{CPL}}$ differs considerably from that of Avron-style hypersequent calculi. In particular, derivations and proofs in $\mathsf{MI}^{\mathsf{CPL}}$ are not trees having hypersequents in their nodes, but sequences of hypersequents. Rules of $\mathsf{MI}^{\mathsf{CPL}}$ transform hypersequents into hypersequents, and a rule is always applied to the last term of a derivation constructed so far. Last but not least, $\mathsf{MI}^{\mathsf{CPL}}$ has no axioms, but comprises rules only.

Given these substantial differences, and taking into account that the concept of hypersequent is loaded with references to Avron-style calculi, let me use a new term for a sequence of sequents. The term chosen is "seqsequent", after Latin *sequentia*, which means (among others) "sequence."[18] No doubt, saying that we aim at a calculus of seqsequents is less misleading than speaking about a calculus of hypersequents. A warning is needed, however. As we will see, the order in which

[17] Starting from the influential paper [1]. Avron-style approach is not the only one, however. A reader interested in different types of hypersequent calculi (including those in which hypersequents are construed as sets or multisets of sequents rather than their sequences) is advised to consult [8], Chapter 4.7.

[18] "Seq" is not a prefix in English, but since many English words begin with prefixes rooted in Latin, I hope that this proposal is acceptable, at least for the purposes of this paper. A reader familiar with programming is kindly requested to suspend any associations he/she may have.

sequents occur in a "seqsequent" does not determine the order of application of rules of the calculus. We are speaking about a calculus of seqsequents only to stress that rules of the calculus operate on "seqsequents", that is, a rule transforms a sequence of sequents into a sequence of sequents.

6.1.1 Numerically Annotated Wffs, Sequents, and Seqsequents

We will be operating with sequents based on sequences of numerically annotated wffs. By a *numerically annotated wff* (na-wff for short) we mean an expression of the form $A^{[i]}$, where A is a wff and i is a numeral from the set $\{1, 2, 3, \ldots\}$. Let us stress that numerals are here proof-theoretic devices only. It is not assumed that they refer to possible worlds or perform the function of labels.

By a *sequent* we will mean an expression of the form:

$$C_1^{[i_1]}, \ldots, C_m^{[i_m]} \vdash \tag{36}$$

where $C_1^{[i_1]}, \ldots, C_m^{[i_m]}$ is a finite sequence of na-wffs; when $m = 0$, we write the corresponding sequent as $\emptyset \vdash$. Although we consider sequents with empty succedents, it is no accident that we put the turnstile $\vdash$ into a sequent. This will allow us to differentiate between operations on sequents and operations on sequences of annotated wffs (see below).

An *atomic sequent* is of the form:

$$l_1^{[i_1]}, \ldots, l_m^{[i_m]} \vdash \tag{37}$$

where $l_1, \ldots, l_m$ are literals, that is, propositional variables or their negations. An atomic sequent (37) is *closed* if it involves na-wffs based on complementary literals, i.e. there exist $l_j^{[i_j]}, l_k^{[i_k]}$ $(1 \leqslant j, k \leqslant m)$ such that $l_j = \neg l_k$. An atomic sequent which is not closed is called *open*.

We use the Greek lower-case letters σ, θ, χ, possibly with subscripts, as metalanguage variables for finite sequences of na-wffs, the empty sequence included.

Let $\sigma \vdash$ be a sequent. We define:

$$\mathsf{wff}(\sigma \vdash) = \{A \in \texttt{Form} : A^{[i]} \textit{ is a term of } \sigma\}.$$

By $f_{\setminus[i_j]}(\sigma)$ we mean the subsequence of σ resulting from it by removing all its terms (i.e. na-wffs) which are annotated with the numeral i_j. Needless to say, $f_{\setminus[i_j]}(\sigma) \vdash$ is a sequent.[19]

[19] If i_j is the only numeral which occurs in na-wffs of σ, then $f_{\setminus[i_j]}(\sigma) \vdash$ equals $\emptyset \vdash$, which is, by definition, a sequent. Of course, $\mathsf{wff}(\emptyset \vdash) = \emptyset$.

A *seqsequent* is a finite sequence of sequents. We use the Greek upper-case letters Φ, Ψ, Θ, with subscripts when necessary, as metalinguistic variables for seqsequents. By a *constituent* of a seqsequent we mean any sequent which is a term of the seqsequent.

Finally, we distinguish *ordered sequents*.

An ordered sequent is a sequent which falls under the schema:

$$C_1^{[1]}, \ldots, C_m^{[m]} \vdash \tag{38}$$

where $m \geqslant 1$, and $C_1, \ldots, C_m$ are pairwise syntactically distinct wffs when $m > 1$. Thus, besides sequents of the form $A^{[1]} \vdash$, ordered sequents are sequents whose consecutive terms (with the exception of the turnstile), are pairwise syntactically distinct wffs annotated with consecutive numerals (occurring in curly brackets), starting from the numeral 1.[20] At the metalanguage level, ordered sequents of the form (38) will be concisely written as:

$$C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash \tag{39}$$

As we will see, in order to show that $\{C_1, \ldots, C_m\}$ is a MI-set it suffices to prove the corresponding ordered sequent $C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash$. Moreover, having a disproof of the ordered sequent is tantamount to showing that the corresponding set of wffs, albeit inconsistent, is not a MI-set.

6.1.2 Rules and Proofs

In order to present the rules of $\mathsf{MI}^{\mathsf{CPL}}$ in a concise manner let us introduce some notational conventions first.

Following [22], we distinguish between α-wffs and β-wffs, and we assign two further wffs to any of them, in the way presented in Table 3.

We use the sign $'$ as the concatenation-sign for sequences of na-wffs. For brevity, we assume that a metalanguage expression of the form $\sigma\, ' A^{[i]}$ denotes the concatenation of sequence σ and the one-term sequence $\langle A^{[i]} \rangle$, while a metalanguage expression of the form $\sigma\, ' A^{[i]}\, '\, \theta$ refers to the concatenation of sequence $\sigma\, ' A^{[i]}$ and sequence θ.

The semicolon will perform the role of the concatenation-sign for seqsequents. We usually omit angle brackets when referring to a seqsequent which has only one constituent. Thus $\Psi;\ \sigma \vdash$ stands for the concatenation of Ψ and $\langle \sigma \vdash \rangle$. The expression $\Psi\ ; \sigma \vdash;\ \Phi$ refers to the concatenation of $\Psi;\ \sigma \vdash$ and Φ.

[20] Each ordered sequent is a sequent, but not the other way round. For instance, the expressions $p^{[4]}, q^{[2]} \vdash$ and $p^{[1]}, p^{[3]} \vdash$ are sequents in our sense, but none of them is an ordered sequent. Similarly, $p^{[1]}, p^{[1]} \vdash$ and $p^{[1]}, p^{[2]} \vdash$ are sequents, though neither of them is an ordered sequent.

α	α_1	α_2	β	β_1	β_2
$A \wedge B$	A	B	$\neg(A \wedge B)$	$\neg A$	$\neg B$
$\neg(A \vee B)$	$\neg A$	$\neg B$	$A \vee B$	A	B
$\neg(A \rightarrow B)$	A	$\neg B$	$A \rightarrow B$	$\neg A$	B

Table 3: *α-wffs and β-wffs*

The calculus $\mathsf{MI}^{\mathsf{CPL}}$ has only rules which operate on seqsequents. No axioms are provided. Here are the primary rules of $\mathsf{MI}^{\mathsf{CPL}}$:

$$\mathsf{R}_{\alpha}^{[i]}: \quad \frac{\Phi;\ \sigma \ ' \ \alpha^{[i]} \ ' \ \theta\ \vdash; \Psi}{\Phi;\ \sigma \ ' \ \alpha_1^{[i]} \ ' \ \alpha_2^{[i]} \ ' \ \theta \vdash; \Psi}$$

$$\mathsf{R}_{\beta}^{[i]}: \quad \frac{\Phi;\ \sigma \ ' \ \beta^{[i]} \ ' \ \theta \vdash; \Psi}{\Phi;\ \sigma \ ' \ \beta_1^{[i]} \ ' \ \theta \vdash;\ \sigma \ ' \ \beta_2^{[i]} \ ' \ \theta \vdash; \Psi}$$

$$\mathsf{R}_{\neg\neg}^{[i]}: \quad \frac{\Phi;\ \sigma \ ' \ \neg\neg A^{[i]} \ ' \ \theta \vdash; \Psi}{\Phi;\ \sigma \ ' \ A^{[i]} \ ' \ \theta \vdash; \Psi}$$

Any of Φ, Ψ, σ, θ can be empty.

Observe that rules of $\mathsf{MI}^{\mathsf{CPL}}$ "act locally": if a rule is applied to a seqsequent, only one constituent and only one occurrence of a na-wff in the constituent are acted upon, while the other occurrences and other constituents remain unaffected. Moreover, any new na-wff that comes into play due to an application of a rule is annotated with the same numeral as the na-wff acted upon.

We are now ready for an introduction of the concept of proof.

Definition 7 (Proof)**.** *A finite sequence of seqsequents $\Theta_1, \ldots, \Theta_n$ is a $\mathsf{MI}^{\mathsf{CPL}}$-proof of an ordered sequent $C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash$ iff*

1. $\Theta_1 = \langle C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash \rangle$,
2. *Θ_{j+1} results from Θ_j by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, where $1 \leqslant j < n$,*
3. *each constituent of Θ_n is a closed atomic sequent,*
4. *for each $k \in \{1, \ldots, m\}$ there exists a constituent $\sigma \vdash$ of Θ_n such that the sequent $f_{\setminus[k]}(\sigma) \vdash$ is an open atomic sequent or is of the form $\emptyset \vdash$.*

An ordered sequent is provable in $\mathsf{MI}^{\mathsf{CPL}}$ *iff the sequent has a* $\mathsf{MI}^{\mathsf{CPL}}$*-proof.*

Remark 7. The concept of proof introduced above is non-standard in many respects. First, a proof is a sequence of seqsequents. Second, notice that it is an ordered sequent (i.e. a sequent in which wffs occurring left of the turnstile are annotated with consecutive numerals, starting from 1) that performs the role of an "input" of a proof: the first line of a proof is a one-term seqsequent involving an ordered sequent. We do not introduce the concept of proof of a sequent in general, but only of an ordered sequent. As we will see, this is sufficient for our purposes. Third, proofs in $\mathsf{MI}^{\mathsf{CPL}}$ are strictly linear: Θ_{j+1} results by a rule from Θ_j only. Clauses (3) and (4) of the definition jointly ensure that $\mathsf{wff}(C^{[\overrightarrow{m}]}_{\overrightarrow{m}} \vdash)$ is a MI-set.

Provability in $\mathsf{MI}^{\mathsf{CPL}}$ and the property of being a MI-set are linked in a way characterized by:

Theorem 12 (Soundness w.r.t. MI-sets)**.** *Let X be a finite non-empty set of wffs, and let $\sigma \vdash$ be an ordered sequent such that* $\mathsf{wff}(\sigma \vdash) = X$*. If the sequent $\sigma \vdash$ is provable in* $\mathsf{MI}^{\mathsf{CPL}}$*, then X is a* MI*-set.*

A proof of Theorem 12 is presented in the Appendix.

Due to Theorem 12, in order to show that X is a MI-set it suffices to prove an ordered sequent $C^{[\overrightarrow{m}]}_{\overrightarrow{m}} \vdash$ for which the equation $X = \mathsf{wff}(C^{[\overrightarrow{m}]}_{\overrightarrow{m}})$ holds.

Example 1. $\{p, \neg p\}$ *is a* MI*-set.*

The one-term sequence $\langle p^{[1]}, \neg p^{[2]} \vdash \rangle$ is a proof of the sequent $p^{[1]}, \neg p^{[2]} \vdash$, since $f_{\backslash [1]}(p^{[1]}, \neg p^{[2]}) \vdash = \neg p^{[2]} \vdash$ and $f_{\backslash [2]}(p^{[1]}, \neg p^{[2]}) \vdash = p^{[1]} \vdash$.

For brevity, in what follows we will be omitting angle brackets in the first line of a proof, and in the case of one-term seqsequents.

Example 2. $\{p \wedge \neg p\}$ *is a* MI*-set.*

Here is a proof of the sequent $(p \wedge \neg p)^{[1]} \vdash$ (inscriptions of the form $\mathsf{R}^{[i]}_x$ do not belong to proofs, but indicate what rule has been applied to the seqsequent which occurs on the left):

$$\begin{array}{ll} (p \wedge \neg p)^{[1]} \vdash & \qquad \mathsf{R}^{[1]}_{\alpha} \\ p^{[1]}, \neg p^{[1]} \vdash & \end{array}$$

Notice that $f_{\backslash [1]}(p^{[1]}, \neg p^{[1]} \vdash)$ equals $\emptyset \vdash$.

Example 3. $\{p, \neg(\neg p \to q)\}$ *is a* MI*-set.*

The following is a proof of a corresponding ordered sequent:

$$p^{[1]}, \neg(\neg p \to q)^{[2]} \vdash \qquad \mathsf{R}_{\alpha}^{[2]}$$
$$p^{[1]}, \neg p^{[2]}, \neg q^{[2]} \vdash$$

For the sake of transparency, from now on we highlight the na-wff on which the rule indicated to the right acts upon. We tick exemplary occurrences of numerals due to which clause (4) of the definition of proof is satisfied.

Example 4. $\{\neg p, \neg q, p \vee q\}$ *is a* MI*-set.*

Here is a proof of a corresponding ordered sequent:

$$\neg p^{[1]}, \neg q^{[2]}, (p \vee q)^{[3]} \vdash \qquad \mathsf{R}_{\beta}^{[3]}$$
$$\neg p^{[1\checkmark]}, \neg q^{[2]}, p^{[3\checkmark]} \vdash; \neg p^{[1]}, \neg q^{[2\checkmark]}, q^{[3]} \vdash$$

Example 5. $\{p \vee (q \vee r), \neg(p \vee q), \neg(p \vee r)\}$ *is a* MI*-set.*

$$p \vee (q \vee r))^{[1]}, \neg(p \vee q)^{[2]}, \neg(p \vee r)^{[3]} \vdash \qquad \mathsf{R}_{\alpha}^{[3]}$$
$$(p \vee (q \vee r))^{[1]}, \neg(p \vee q)^{[2]}, \neg p^{[3]}, \neg r^{[3]} \vdash \qquad \mathsf{R}_{\alpha}^{[2]}$$
$$(p \vee (q \vee r))^{[1]}, \neg p^{[2]}, \neg q^{[2]}, \neg p^{[3]}, \neg r^{[3]} \vdash \qquad \mathsf{R}_{\beta}^{[1]}$$
$$p^{[1]}, \neg p^{[2]}, \neg q^{[2]}, \neg p^{[3]}, \neg r^{[3]} \vdash; (q \vee r)^{[1]}, \neg p^{[2]}, \neg q^{[2]}, \neg p^{[3]}, \neg r^{[3]} \vdash \qquad \mathsf{R}_{\beta}^{[1]}$$
$$p^{[1\checkmark]}, \neg p^{[2]}, \neg q^{[2]}, \neg p^{[3]}, \neg r^{[3]} \vdash; q^{[1]}, \neg p^{[2]}, \neg q^{[2\checkmark]}, \neg p^{[3]}, \neg r^{[3]} \vdash;$$
$$r^{[1]}, \neg p^{[2]}, \neg q^{[2]}, \neg p^{[3]}, \neg r^{[3\checkmark]} \vdash$$

Example 6. $\{p \to (q \to r), p \to q, \neg(p \to r)\}$ *is a* MI*-set.*

$$(p \to (q \to r))^{[1]}, (p \to q)^{[2]}, \neg(p \to r)^{[3]} \vdash \qquad \mathsf{R}_{\alpha}^{[3]}$$
$$(p \to (g \to r))^{[1]}, (p \to q)^{[2]}, p^{[3]}, \neg r^{[3]} \vdash \qquad \mathsf{R}_{\beta}^{[2]}$$
$$(p \to (q \to r))^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash; (p \to (q \to r))^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash \qquad \mathsf{R}_{\beta}^{[1]}$$
$$\neg p^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash; (q \to r)^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash;$$
$$(p \to (q \to r))^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash \qquad \mathsf{R}_{\beta}^{[1]}$$
$$\neg p^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash; \neg q^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash; r^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash;$$
$$(p \to (q \to r))^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash \qquad \mathsf{R}_{\beta}^{[1]}$$
$$\neg p^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash; \neg q^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash; r^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash;$$
$$\neg p^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash; (q \to r)^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash \qquad \mathsf{R}_{\beta}^{[1]}$$
$$\neg p^{[1]}, \neg p^{[2]}, p^{[3\checkmark]}, \neg r^{[3]} \vdash; \neg q^{[1]}, \neg p^{[2\checkmark]}, p^{[3]}, \neg r^{[3]} \vdash; r^{[1]}, \neg p^{[2]}, p^{[3]}, \neg r^{[3]} \vdash;$$

$\neg p^{[1\checkmark]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash; \neg q^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash; r^{[1]}, q^{[2]}, p^{[3]}, \neg r^{[3]} \vdash$

The system $\mathsf{MI}^{\mathsf{CPL}}$ is complete w.r.t. MI-sets.

Theorem 13 (Completeness w.r.t. MI-sets). *If X is a* MI*-set, then any ordered sequent $\sigma \vdash$ such that* $\mathsf{wff}(\sigma \vdash) = X$ *is provable in* $\mathsf{MI}^{\mathsf{CPL}}$*.*

A proof of Theorem 13 is presented in the Appendix.

6.1.3 Disproofs

The system $\mathsf{MI}^{\mathsf{CPL}}$ is useful not only in showing that something is a MI-set, but also in demonstrating that a set of wffs is inconsistent yet not minimally so. The latter can be achieved by providing a *disproof* of an ordered sequent which corresponds to the set of wffs under consideration.

Definition 8 (Disproof). *A finite sequence of seqsequents $\Theta_1, \ldots, \Theta_n$ is a $\mathsf{MI}^{\mathsf{CPL}}$-disproof of an ordered sequent $C_{\vec{m}}^{[\vec{m}]} \vdash$ iff*

1. $\Theta_1 = \langle C_{\vec{m}}^{[\vec{m}]} \vdash \rangle$,
2. *Θ_{j+1} results from Θ_j by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, where $1 \leqslant j < n$,*
3. *each constituent of Θ_n is a closed atomic sequent,*
4. *there exists $k \in \{1, \ldots, m\}$ such that for each constituent $\sigma \vdash$ of Θ_n, the sequent $f_{\setminus[k]}(\sigma) \vdash$ is closed.*

An ordered sequent is disprovable in $\mathsf{MI}^{\mathsf{CPL}}$ iff the sequent has a $\mathsf{MI}^{\mathsf{CPL}}$-disproof.

Observe that proofs and disproofs differ only with respect to their closing conditions. To be more precise, each sequence of seqsequents $\Theta_1, \ldots, \Theta_n$ satisfying the clauses 1, 2, and 3 of the definition of proof (i.e. Definition 7) and violating clause 4 of the definition is not a proof, but a disproof.

The following holds (for a proof, see the Appendix):

Theorem 14. *If there exists a $\mathsf{MI}^{\mathsf{CPL}}$-disproof of an ordered sequent $C_{\vec{m}}^{[\vec{m}]} \vdash$, then the set $\mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash)$ is inconsistent, but is not a* MI*-set.*

Here is an example of a disproof of $(p \rightarrow q)^{[1]}, p^{[2]}, \neg(p \vee q)^{[3]} \vdash$ (clause 4 is satisfied w.r.t. the numeral 1):

Example 7. $\boxed{(p \to q)^{[1]}}, p^{[2]}, \neg(p \vee q)^{[3]} \vdash$ $\qquad\mathsf{R}_{\beta}^{[2]}$

$\neg p^{[1]}, p^{[2]}, \boxed{\neg(p \vee q)^{[3]}} \vdash; q^{[1]}, p^{[2]}, \neg(p \vee q)^{[3]} \vdash$ $\qquad\mathsf{R}_{\alpha}^{[2]}$

$\neg p^{[1]}, p^{[2]}, \neg p^{[3]}, \neg q^{[3]} \vdash; q^{[1]}, p^{[2]}, \boxed{\neg(p \vee q)^{[3]}} \vdash$ $\qquad\mathsf{R}_{\alpha}^{[2]}$

$\neg p^{[1]}, p^{[2]}, \neg p^{[3]}, \neg q^{[3]} \vdash; q^{[1]}, p^{[2]}, \neg p^{[3]}, \neg q^{[3]} \vdash$

Thus $\{p \to q, p, \neg(p \vee q)\}$, though inconsistent, is not a MI-set.

Finally, the following holds as well:

Theorem 15. *If X is a finite inconsistent set of wffs which is not a* MI*-set, then any ordered sequent $\sigma \vdash$ such that* $\mathsf{wff}(\sigma \vdash) = X$ *is disprovable in* $\mathsf{MI}^{\mathsf{CPL}}$.

For a proof of Theorem 15 see the Appendix.

Remark 8. The primary rules of $\mathsf{MI}^{\mathsf{CPL}}$ transform wffs inside sequents analogously as Smullyan's tableaux rules do. It is possible to build a calculus of MI-sets in the "standard" tableau format, with rules defined as operating directly on (annotated) wffs, while occurrences of these wffs are nodes of respective trees. This would require adding an annotation mechanism and specifying new closing conditions. The format of $\mathsf{MI}^{\mathsf{CPL}}$ is akin to that of the so-called erotetic calculi (cf., e.g.,[27], [11],[10]). The main difference lies in the fact that rules of $\mathsf{MI}^{\mathsf{CPL}}$ operate on sequences of sequents, while rules of erotetic calculi act upon questions based on sequences of sequents. Moreover, annotations are exploited here in a new manner, and closing conditions of a proof are more demanding. The advantage of the current format over the "standard" tableaux approach lies in its relative simplicity at the metatheoretical level. Moreover, it is known that proofs written in the erotetic calculi format can be transformed into proofs in tableaux calculi (cf.[10]), sequent calculi (cf. [11]) or even Hilbert-style calculi (cf. [7]). These effects do not disappear when we move from questions based on sequences of sequents to the "inner" sequences of sequents.

6.2 Soundness and Completeness of $\mathsf{MI}^{\mathsf{CPL}}$ w.r.t. Strong Entailments

As we have shown, the calculus $\mathsf{MI}^{\mathsf{CPL}}$ is sound and complete w.r.t. MI-sets. Due to Theorem 5, the fact that $X, \neg Y$ is a MI-set guarantees that $X \mathrel{\|\!\!\prec} Y$ holds provided that $X \cap \neg Y = \emptyset$ is the case. So when we restrict ourselves to ordered sequents built in such a way that the fulfilment of the latter condition is secured, proofs of these ordered sequents can be viewed as demonstrations that strong mc-entailment hold in the cases considered.

Theorem 16 (Soundness w.r.t. strong mc-entailment)**.** *Let* $X = \{A_1, \ldots, A_n\}$ *and* $Y = \{B_1, \ldots, B_k\}$*, where* $n + k > 0$ *and* $A_i \neq \neg B_j$ *for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, k$*. If the ordered sequent:*

$$A_1^{[1]}, \ldots, A_n^{[n]}, \neg B_1^{[n+1]}, \ldots, \neg B_k^{[n+k]} \vdash$$

is provable in $\mathsf{MI}^{\mathsf{CPL}}$*, then* $X \Vert\!\prec Y$*.*

Proof. By Theorem 5 and Theorem 12. □

Theorem 17 (Completeness w.r.t. strong mc-entailment)**.** *Let* $X = \{A_1, \ldots, A_n\}$ *and* $Y = \{B_1, \ldots, B_k\}$*, where* $n + k > 0$ *and* $A_i \neq \neg B_j$ *for* $i = 1, \ldots, n$ *and* $j = 1, \ldots, k$*. If* $X \Vert\!\prec Y$*, then the ordered sequent:*

$$A_1^{[1]}, \ldots, A_n^{[n]}, \neg B_1^{[n+1]}, \ldots, \neg B_k^{[n+k]} \vdash$$

is provable in $\mathsf{MI}^{\mathsf{CPL}}$*.*

Proof. By Theorem 5 and Theorem 13. □

As for strong sc-entailment, one gets analogous results by applying Theorem 8 instead of Theorem 5.

7 Some Conceptual Applications

7.1 Deep Contraction

Let us imagine that we are working with a consistent non-empty set of CPL-wffs X (for instance, representing a database or a belief base) and that a contingent CPL-wff B has been derived from X. Assume that the derivation mechanism used preserves CPL-entailment. Now suppose that we have strong, though independent from X, reasons to believe that $\neg B$ rather than B is the case. As long as we stick to Classical Logic, extending X with $\neg B$ is not a good move. An option is to switch to some non-monotonic logic and its consequence operation. As we have shown, strong sc-entailment is not monotone. But no extension of X produces $\neg B$ as a strongly sc-entailed consequence of X. This is due to:

Corollary 38. *If* $X \models\!\prec B$*, then there is no proper superset* Z *of* X *such that* $Z \models\!\prec \neg B$*.*

Proof. Let $X \models\!\prec B$. Suppose that $Z \models\!\prec \neg B$, where $X \subseteq Z$ and $X \neq Z$. If $X \models\!\prec B$, then $X \models B$ and hence $Z \models B$. If $Z \models\!\prec \neg B$, then $Z \models \neg B$. Therefore Z is inconsistent and thus, by Corollary 14, $Z \not\models\!\prec \neg B$. □

A rational move is to contract X first, and in a way that prevents the appearance of B as a conclusion of any legitimate (i.e. preserving classical entailment) derivation from the contracted set. How to achieve this? One can examine the derivation of B from X that has actually been performed, identify the elements of X used as premises, and then contract X by removing from it at least one wff which was used as a premise in the performed derivation. This, however, will not do: it is possible that B is classically entailed by many subsets of X, including some that do not contain the just removed wff(s), and thus B can still be legitimately derived from the set contracted in the above manner. Examining all possible legitimate derivations of B from X constitutes a difficult if not a hopeless task. However, a solution is suggested by the content of Theorem 11. By and large, it suffices to consider all the finite subsets of X that strongly sc-entail B, and to remove from X exactly one element of every such subset. A contracted set obtained in this way does not CPL-entail the wff B and therefore no legitimate derivation leads from the set to B.

Remark 9. The way of proceeding proposed above is akin to (but not identical with) the well-known idea of consistency restoring by calculating a minimal hitting set of the family of all minimally inconsistent subsets of an inconsistent set in order to eliminate elements of the hitting set from the inconsistent set in question.[21] The general idea goes back to [15] and gave rise to some related constructions.[22] However, contraction of the analysed kind does not aim at consistency restoring, but at making a legitimate deduction of B from the resultant set impossible. These are interconnected, but yet different issues.

In what follows we apply some conceptual tools taken from [29].

Let $\mathbb{F}$ be a family of sets, i.e. a set of sets. These sets need not be disjoint. In the first step we define a related family of pairwise disjoints sets.

Definition 9. $\mathbb{F}^{\otimes} =_{df} \{X^{\otimes} : X \in \mathbb{F}\}$, *where:*

$$X^{\otimes} = \begin{cases} X \times \{X\} & \text{if } X \neq \emptyset, \\ \emptyset & \text{if } X = \emptyset. \end{cases}$$

Since the elements of $\mathbb{F}^{\otimes}$ are pairwise disjoint, by the Axiom of Choice we get:

[21] A set X is a hitting set of a family of sets $\mathbb{F}$ iff $X \cap Y \neq \emptyset$ for each $Y \in \mathbb{F}$. A hitting set of $\mathbb{F}$ is minimal if no proper subset of it is a hitting set of $\mathbb{F}$. Hitting sets are also called choice sets. For hitting/choice sets see, e.g., [24], pp. 335– 338.

[22] Cf. e.g., [3].

Corollary 39. *If $\mathbb{F}^{\otimes} \neq \emptyset$ and $\emptyset \notin \mathbb{F}^{\otimes}$, then there exists a set γ such that γ comprises exactly one element, $\langle A, X \rangle$, of each $X^{\otimes} \in \mathbb{F}^{\otimes}$.*

We introduce an auxiliary notion.

Definition 10. *γ is a $\chi^{\otimes}(\mathbb{F})$-set iff*

1. *$\gamma \subseteq \bigcup \mathbb{F}^{\otimes}$ and*
2. *for each $X^{\otimes} \in \mathbb{F}^{\otimes}$ such that $X^{\otimes} \neq \emptyset$ there exists exactly one $\langle A, X \rangle \in X^{\otimes}$ such that $\langle A, X \rangle \in \gamma$.*

A $\chi^{\otimes}(\mathbb{F})$-set is a set of ordered pairs. We take into account the first projection of the set.

Definition 11. *Let γ be a $\chi^{\otimes}(\mathbb{F})$-set.*

$$\gamma^1 =_{df} \{A : \langle A, X \rangle \in \gamma\}.$$

Now we are able to introduce the crucial technical notion.

Definition 12. *Z is a $\chi(\mathbb{F})$-set iff $Z = \gamma^1$ for some $\chi^{\otimes}(\mathbb{F})$-set γ.*

By and large, a $\chi(\mathbb{F})$-set is a set comprising exactly one *representative*, with regard to the above construction, of each non-empty set belonging to a family of sets $\mathbb{F}$. The representatives of distinct sets in a χ-set need not be distinct. One should not confuse the existence of exactly one representative (of the above kind) of each set belonging to a family of sets with the existence of a system of distinct representatives of the family.[23] One can prove that a χ-set always exists, i.e. for any family of sets $\mathbb{F}$ there exists a $\chi(\mathbb{F})$-set (cf. [29]).

Let us now come back to the contraction issue. The following holds.

Theorem 18 (Deep contraction)**.** *Let X be a consistent non-empty set of wffs, and let B be a non-valid wff such that $X \models B$. Let $\mathbb{F} = \{W \subseteq X : W \models\!\!\!< B\}$, and let Z be a $\chi(\mathbb{F})$-set. Then $(X \setminus Z) \not\models B$.*

Proof. By Theorem 11, the family $\mathbb{F}$ is non-empty. If B is non-valid, $\emptyset \notin \mathbb{F}$. Thus $X' \neq \emptyset$ for each $X' \in \mathbb{F}$, and hence $Z \neq \emptyset$.

The set $X \setminus Z$ is consistent, since X is, by assumption, consistent.

Suppose that $(X \setminus Z) \models B$. It follows that $(X \setminus Z) \neq \emptyset$ (as B is not valid) and, by Theorem 11, that $Y \models\!\!\!< B$ for some finite subset Y of $X \setminus Z$. Moreover, $Y \neq \emptyset$;

[23] As it is well-known, a system of distinct representatives – a transversal of a family of sets – does not always exist; cf., e.g., [26], Chapter 8.

otherwise B would have been valid. But the only subsets of X that strongly sc-entail B are the sets in $\mathbb{F}$. Hence $Y = X^\circ$ for some element, X°, of $\mathbb{F}$. But $(X' \cap Z) \neq \emptyset$ for each $X' \in \mathbb{F}$. Hence $(X^\circ \cap Z) \neq \emptyset$. On the other hand, $(Y \cap Z) = \emptyset$ due to the fact that Y is a subset of $X \setminus Z$. It follows that $Y \neq X^\circ$. We arrive at a contradiction. Therefore $(X \setminus Z) \not\models B$. □

Let us stress that Theorem 18 speaks about *any* χ-set of the family of subsets of X which strongly sc-entail B. There are usually many such sets. Each of them may be subtracted from X in order to arrive at a subset of X that does not (classically) entail B. In other words, "deep contraction" can be successfully performed in many ways and its outcome depends on the χ-set chosen.[24] As for the multiplicity of possible outcomes, and their dependence on factors different from the set subjected to be contracted and the wff w.r.t. which the operation is performed, deep contraction does not differ from other contraction operations characterized in belief revision theories. Note, however, that deep contraction has a kind of computational flavour. In order to perform it one needs a χ-set of the family of subsets of X which strongly sc-entail B, and this requires that the family has to be "calculated" first. Given the content of Theorem 5, this, in turn, can be achieved by identifying all the minimally inconsistent subsets of an inconsistent set of some kind.[25] Algorithms for solving such problems are already known in the literature.[26]

Remark 10. A set of wffs X supposed to be contracted w.r.t. B may be either finite or infinite. In the latter case it can happen that the family of subsets of X that sc-entail B is countably or even uncountably infinite. It follows that the relevant χ-sets may be infinite. However, we are dealing here with Classical Logic, in which entailment is compact: everything entailed by an infinite set of wffs is also entailed by some finite subset(s) of the set. One can easily prove:

[24] A simple example may be of help. Let $X = \{p \vee q \rightarrow r, p, q\}$ and $B = r$. The relevant family of MI-sets comprises $\{p \vee q \rightarrow r\}$ and $\{p \vee q \rightarrow r, q\}$; let us designate it by $\mathbb{F}$. The $\chi(\mathbb{F})$-sets are: (1) $\{p, q\}$, (2) $\{p \vee q \rightarrow r\}$, (3) $\{p \vee q \rightarrow r, q\}$, (4) $\{p \vee q \rightarrow r, p\}$. The result of deep contraction of X, depending on the $\chi(\mathbb{F})$-set used, is (1') $\{p \vee q \rightarrow r\}$, or (2') $\{p, q\}$, or (3') $\{p\}$, or (4') $\{q\}$. Which $\chi(\mathbb{F})$-set is to be used depends on epistemic factors. By the way, the example presented above shows that deep contraction does not amount to subtracting a minimal choice set of the family of all MI-sets in question.

Belief revision theories view contraction as an operation which is supposed to achieve its goal(s) in an "economical" manner: the loss should be kept to a minimum. This means many things, depending on an account advocated. As for deep contraction, the "minimalization of loss" issue is only of a secondary importance.

[25] More specifically, all minimally inconsistent subsets of $X \cup \{\neg B\}$ such that $\neg B$ belongs to each of them have to be identified first. Then the family $\{Y \subseteq X : Y \cup \{\neg B\}$ is a MI-set and '$\neg B$' $\notin Y\}$ constitutes the solution.

[26] See, e.g., [12], and [3].

Corollary 40. *Let X be an infinite consistent set of wffs, and let B be a non-valid wff such that $X \models B$. If Y is a finite subset of X such that $Y \models B$, and Z is a $\chi(\mathbb{F}^*)$-set, where $\mathbb{F}^* = \{W \subseteq Y : W \Vdash\!\!< B\}$, then $X \cap (Y \setminus Z) \not\models B$.*

Proof. Suppose otherwise. Then $(Y \setminus Z) \models B$, contrary to Theorem 18. □

Thus when entailment is compact, an infinite set X can also be "deeply contracted" w.r.t. B without relying on infinite χ-set(s) that correspond(s), in the way described above, to the whole X. It suffices to use a χ-set which corresponds to a finite subset Y of X that classically entails B. Needless to say, the resultant set $X \cap (Y \setminus Z)$ will be finite.

7.2 Argument Analysis

7.2.1 Strong Entailments and Lehrer's Notion of Relevant Deductive Argument

Strong sc-entailment is a special case of strong mc-entailment. There are, however, close affinities between the concept of strong sc-entailment and the notion of relevant deductive argument introduced long ago by Keith Lehrer (cf. [9]). Here is Lehrer's definition:

> An argument RD is a relevant deductive argument if and only if RD contains a non-empty set of premises P_1, P_2, ..., P_n and a conclusion C such that a set of statements consisting of just $P_1, P_2, \ldots, P_n$, and $\neg C$ (or any truth functional equivalent of $\neg C$) is a minimally inconsistent set. A set of statements is a minimally inconsistent set if and only if the set of statements is logically inconsistent and such that every proper subset of the set is logically consistent. ([9], p. 298.)

Is strong sc-entailment just the semantic relation that holds between premises and conclusions of Lehrer's relevant deductive arguments? As Theorem 8 illustrates, the fact that $\{P_1, \ldots, P_n, \neg C\}$ is a MI-set is a necessary but insufficient condition for $\{P_1, P_2, \ldots, P_n\} \Vdash\!\!< C$ to hold. It is additionally required that '$\neg C$' $\notin \{P_1, P_2, \ldots, P_n\}$. On the other hand, the statement "a set of statements consisting of just P_1, P_2, ..., P_n, and $\neg C$ (or any truth functional equivalent of $\neg C$)" seems to secure that the additional requirement is to be met. Anyway, relevant deductive arguments in Lehrer's sense (henceforth: *rd-arguments*) and deductive arguments in which the conclusion is *strongly* sc-entailed by the premises – let us refer to them as to $\Vdash\!\!<$-arguments – share basic properties. In both cases, as stipulated by Lehrer and witnessed by corollaries 22 and 16, only contingent wffs can serve as premises and conclusions, and the set of premises is always consistent. Assuming that a set of premises of an argument must be non-empty, there is neither rd-argument nor $|\!\!\prec$-argument which leads to a valid wff or to a contradictory/inconsistent conclusion (cf. corollaries 20 and 17, respectively). So some classes of relevant (in the intuitive sense of the word) deductive arguments are beyond the scope of either analysis. On the other hand, our considerations, though indirectly, throw new light of the properties of rd-arguments. Theorem 11 yields that for each deductive argument $\mathcal{A}$ from a consistent set of premises there exists a corresponding $|\!\!\prec$-argument leading from a finite subset of the set of premises of $\mathcal{A}$ to the conclusion of the argument $\mathcal{A}$. The premises of an $|\!\!\prec$-argument are mutually independent (cf. Corollary 25), and their conclusions always share variable(s) with the premises (cf. Corollary 24). A multi-premise $|\!\!\prec$-argument differs from the respective single-premise $|\!\!\prec$-argument based on a conjunction of premises of the multi-premise argument (cf. section 4.2.3).

It seems that the intuitive concept of *linked* multi-premise deductive argument can be successfully explicated in terms of strong sc-entailment: a linked multi-premise deductive argument is an argument whose conclusion is strongly sc-entailed by the set of premises.

7.2.2 Multiple-Conclusion Arguments?

The concept of argument is sometimes generalized to include arguments containing a finite number of conclusions. As a result, one gets an unproblematic class of arguments having exactly one conclusion – let us call them *sc-arguments* – and a problematic class of arguments having at least two (but still finitely many) conclusions. Let us call the latter *mc-arguments*.

As we observed, strong sc-entailment is, in principle, the semantic relation which holds between premises and conclusions of relevant (in the Lehrer's sense) deductive sc-arguments. By analogy, strong mc-entailment can be viewed as singling out an important class of mc-arguments. We coin them (for the lack of a better idea) *germane mc-arguments*. To be more precise, by a germane mc-argument we mean an mc-argument whose premises strongly mc-entail the *set* of conclusions of the argument. The properties of strong mc-entailment pointed out above seem to speak in favour of this proposal.

There is an ongoing discussion as to whether mc-arguments are artifacts (cf. [17], [23], [2]). An mc-argument is often identified with the corresponding sc-argument whose conclusion is a disjunction of all the "conclusions" of the respective mc-argument. Without pretending to resolve the issue, let us only notice the following.

Consider:

$$\frac{p}{p \vee q} \tag{40}$$

and

$$\frac{p}{p, q} \tag{41}$$

Since $p \mid\!\prec p \vee q$ holds, (40) constitutes an $\mid\!\prec$-argument. But $p \parallel\!\prec \{p, q\}$ does not hold and thus (41) is not a germane mc-argument. So it happens that, having premises fixed, there exist $\mid\!\prec$-arguments leading from the premises(s) to a disjunction, but there is no germane mc-argument that leads from the premise(s) to the set of disjuncts.

Now let us consider:

$$\frac{p \vee q}{p, q} \tag{42}$$

and

$$\frac{p \vee q}{p} \tag{43}$$

$$\frac{p \vee q}{q} \tag{44}$$

(42) is a germane mc-argument, while (43) and (44) are not $\mid\!\prec$-arguments. This is not an exception, but a rule. Due to Corollary 2, if an mc-argument leading from a disjunction to the set of disjuncts is germane, then there is no $\mid\!\prec$-argument that leads from the disjunction to a single disjunct. In general, the existence of a germane mc-argument from X to Y excludes the existence of an $\mid\!\prec$-argument from X to a single conclusion that belongs to Y. Similarly, the existence of an $\mid\!\prec$-argument leading from X to a wff which is only one of the elements of a set of wffs Y suppresses the existence of a germane mc-argument leading from X to Y.

8 Final Remarks

8.1 The First-Order Case

So far we have dealt with the classical propositional case. So a natural question arises: what, if anything, will change when we move to the first-order level and consider strong entailments based on First-Order Logic (FOL)?

As it is well-known, sc-entailment in FOL can be defined either in terms of satisfaction or in terms of truth, and similarly for mc-entailment. However, truth of a wff in a FOL-model equals satisfaction of the wff *under all assignments* of values to individual variables, where the values belong to the universe of the model. Therefore the respective concepts of entailment do not coincide when sentential functions, that is, wffs in which free variables occur, enter the picture, although they coincide on FOL-sentences (i.e. wffs with no free variables). Similarly, inconsistency can be defined either as unsatisfiability or as the lack of a FOL-model which makes true all the wffs in question. These are not the same thing if sentential functions are allowed.[27]

When one wants to move from the propositional level to the first-order one, three possibilities emerge.

The simplest solution is to assume that strong entailments, as well as the other semantic notions employed, are defined for sentences only. The concept of truth under a CPL-valuation is to be replaced with the concept of truth in a FOL-model. Then the results concerning CPL "translate" into the respective results concerning the "sentential part" of FOL. Of course, this does not pertain to results which rely on the assumption that the wffs considered are propositional, in particular to Theorem 4. Needless to say, an analogous remark applies to the other options presented below.

The second option is to allow for sentential functions and to replace "true under a CPL-valuation" with "satisfied in a FOL-model under an assignment of values to individual variables." In such a case inconsistency would mean unsatisfiability. There is, however, a price to be paid. While sc-entailment defined in terms of satisfaction ensures the transmission of truth, mc-entailment defined by means of satisfaction (i.e. roughly, by the clause: "for every assignment ι: if all the wffs in X are satisfied under ι, then at least one wff in Y is satisfied under ι") does not warrant the existence of a *true* wff in Y when all the wffs in X are true. This lack of warranty shows up in the case of mc-entailed sets containing sentential functions. As a consequence, the intuitive meaning of the concept of strong mc-entailment changes.

As for the third option, one allows for sentential functions and replaces "true under a CPL-valuation" with "true in a FOL-model." Now consistency of a set of wffs would mean the existence of a FOL-model which makes all the wffs true. Contingent wffs are these which are true in some, but not all FOL-models. However, sc-entailment of A from X amounts to inconsistency of the set comprising X and *the negation of the universal closure* of A. Similarly, mc-entailment between X and Y holds iff the set $X, \neg\overline{Y}$ is inconsistent, where $\overline{Y}$ is the set of universal closures of elements of Y. So a "translation" of results concerning CPL should be performed with caution. In particular, whenever consistency/inconsistency of propositional formulas of the form $\neg A$ or sets of such formulas have been considered, first-order wffs of the form $\neg\overline{A}$, where $\overline{A}$ is the universal closure of A, should be used. For example, the FOL counterparts of theorems 5 and 8 now are:

$$X \parallel\!\!\prec Y \quad \textit{iff} \quad X \cap \neg\overline{Y} = \emptyset \textit{ and } X, \neg\overline{Y} \textit{ is a } \mathsf{MI}\textit{-set}.$$

[27] For instance, the set $\{P(x), \neg\forall x P(x)\}$, where P is a one-place predicate, is satisfiable, but there is no FOL-model which makes its elements simultaneously true.

$$X \models\!\!\!< B \quad \textit{iff} \quad `\neg\overline{B}\,' \notin X \textit{ and } X, \neg\overline{B} \textit{ is a } \mathsf{MI}\textit{-set.}$$

Another example is this. What we have called "deduction theorems" for strong entailments (cf. theorems 7 and 26), relied upon Corollary 11. However, its counterpart does not hold for FOL when entailments are defined in terms of truth. Instead, we have:

$$Z, \overline{A} \Vdash W \textit{ iff } Z \Vdash \ulcorner \overline{A} \to W \urcorner.$$

As a consequence, in order to get counterparts of theorems 7 and 26 one has to replace A with $\overline{A}$. An analogous remark pertains to corollaries 12, 27, and 28.

8.2 Further Research: Strong Entailments in Non-Classical Logics

In this paper we have concentrated upon Classical Logic. A natural next step is to turn to non-classical logics. Which of the results presented above would remain valid if we defined strong entailments in terms of entailments based on a non-classical logic? No doubt, this is an interesting question. Yet, it deserves a separate paper or even a series of papers. So let me only comment on the relation between the concepts of strong entailments and the concept of minimally inconsistent set. Theorems 5 and 8 (as well as their counterparts for FOL) show how these concepts are interconnected for Classical Logic. However, analogues of theorems 5 and 8 fail in some non-classical logics. Negationless logics provide trivial examples here, but there are others. For instance, in Intuitionistic Logic (INT) the following:

$$\{\neg\neg p, \neg p\} \models_{\mathsf{INT}} \bot$$

$$\{\neg\neg p\} \not\models_{\mathsf{INT}} \bot$$

$$\{\neg p\} \not\models_{\mathsf{INT}} \bot$$

hold and thus $\{\neg\neg p, \neg p\}$ can be regarded as a MI-set. Needless to say, '$\neg p$' $\notin \{\neg\neg p\}$. On the other hand, we have:

$$\neg\neg p \not\models_{\mathsf{INT}} p$$

and hence, assuming that strong sc-entailment presupposes sc-entailment, $\neg\neg p$ and p are not linked with strong sc-entailment. It follows that the "intuitionistic" analogues of theorems 8 and 5 do not hold.

Appendix: Soundness and Completeness of $\mathsf{MI}^{\mathsf{CPL}}$

In order to prove soundness and completeness of the calculus $\mathsf{MI}^{\mathsf{CPL}}$ with respect to MI-sets we need a series of corollaries and lemmas.

As an immediate consequence of definitions introduced in Section 6 we get

Corollary 41. *X is a MI-set iff there exists an ordered sequent $C_{\vec{m}}^{[\vec{m}]} \vdash$ such that $X = \mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash)$ and*

1. *$\mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash)$ is an inconsistent set, and*
2. *for each $k \in \{1, \ldots, m\}$: the set $\mathsf{wff}(f_{\setminus[k]}(C_{\vec{m}}^{[\vec{m}]}) \vdash)$ is consistent.*

The following hold:

Lemma 2.

1. *$\mathsf{wff}(S\ '\ \alpha^{[i]}\ '\ T \vdash)$ is inconsistent iff $\mathsf{wff}(S\ '\ \alpha_1^{[i]}\ '\ \alpha_2^{[i]}\ '\ T \vdash)$ is inconsistent.*
2. *$\mathsf{wff}(S\ '\ \beta^{[i]}\ '\ T \vdash)$ is inconsistent iff $\mathsf{wff}(S\ '\ \beta_1^{[i]}\ '\ T \vdash)$ is inconsistent and $\mathsf{wff}(S\ '\ \beta_2^{[i]}\ '\ T \vdash)$ is inconsistent.*
3. *$\mathsf{wff}(S\ '\ A^{[i]}\ '\ T \vdash)$ is inconsistent iff $\mathsf{wff}(S\ '\ \neg\neg A^{[i]}\ '\ T \vdash)$ is inconsistent.*

Lemma 3.

1. *If $\Phi; \sigma \vdash; \Psi$ results from $\Phi; \theta \vdash; \Psi$ by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, then the set $\mathsf{wff}(\sigma \vdash)$ is inconsistent iff $\mathsf{wff}(\theta \vdash)$ is an inconsistent set.*
2. *If $\Phi; \sigma_1 \vdash; \sigma_2 \vdash; \Theta$ results from $\Phi; \theta \vdash; \Psi$ by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, then both $\mathsf{wff}(\sigma_1 \vdash)$ and $\mathsf{wff}(\sigma_2 \vdash)$ are inconsistent sets iff $\mathsf{wff}(\theta \vdash)$ is an inconsistent set.*

Proof. If $\Phi; \sigma \vdash; \Psi$ results from $\Phi; \theta \vdash; \Psi$ by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, then θ involves a numerically annotated α-wff or a numerically annotated double negated wff. But $X, \alpha \Vdash \emptyset$ iff $X, \alpha_1, \alpha_2 \Vdash \emptyset$, and $X, \neg\neg A \Vdash \emptyset$ iff $X, A \Vdash \emptyset$.

If $\Phi; \sigma_1 \vdash; \sigma_2 \vdash; \Theta$ results from $\Phi; \theta \vdash; \Psi$ by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, then a numerically annotated β-wff is a term of θ. Yet, $X, \beta \Vdash \emptyset$ iff $X, \beta_1 \Vdash \emptyset$ and $X, \beta_2 \Vdash \emptyset$. $\square$

Lemma 4. *If a seqsequent Ψ results from a seqsequent Φ by a rule of $\mathsf{MI}^{\mathsf{CPL}}$, then the following conditions are equivalent:*

1. *for each constituent $\sigma \vdash$ of Φ: the set $\mathsf{wff}(\sigma \vdash)$ is inconsistent,*
2. *for each constituent $\theta \vdash$ of Ψ: the set $\mathsf{wff}(\theta \vdash)$ is inconsistent.*

Proof. By Lemma 3. □

Theorem 12 (Soundness w.r.t. MI-sets)**.** *Let X be a finite non-empty set of wffs, and let $\sigma \vdash$ be an ordered sequent such that* $\mathsf{wff}(\sigma \vdash) = X$. *If the sequent $\sigma \vdash$ is provable in* $\mathsf{MI}^{\mathsf{CPL}}$*, then X is a* MI*-set.*

Proof. Let $C_{\vec{m}}^{[\vec{m}]} \vdash$ be an arbitrary but fixed ordered sequent such that $X = \mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash)$. Assume that

$$\Theta_1, \ldots, \Theta_n \tag{45}$$

is a proof of the sequent $C_{\vec{m}}^{[\vec{m}]} \vdash$ in $\mathsf{MI}^{\mathsf{CPL}}$. By Definition 7, each constituent of Θ_n is a closed atomic sequent. Hence the set $\mathsf{wff}(\theta \vdash)$ is inconsistent for each constituent $\theta \vdash$ of Θ_n. Therefore, by Lemma 4, the set $\mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash)$ is inconsistent, that is, X is inconsistent.

We shall prove the following:

(★) *if Θ_{j+1} has a constituent, $\sigma \vdash$, such that the set $\mathsf{wff}(f_{\backslash[k]}(\sigma) \vdash)$ is consistent, then Θ_j has a constituent, θ, such that the set $\mathsf{wff}(f_{\backslash[k]}(\theta) \vdash)$ is consistent, where $1 \leqslant j < n$ and $1 \leqslant k \leqslant m$.*

Let $\sigma \vdash$ be a constituent of Θ_{j+1} for which the set $\mathsf{wff}(f_{\backslash[k]}(\sigma) \vdash)$ is consistent. Recall that rules of $\mathsf{MI}^{\mathsf{CPL}}$ "act locally": if a rule is applied to a seqsequent, only one constituent and only one occurrence of a na-wff in the constituent are acted upon (more precisely, only one term of the sequence of na-wffs which occurs in the constituent is transformed). When Θ_{j+1} results from Θ_j by a rule, the following cases are possible:

(a) $\sigma \vdash$ has been rewritten from Θ_j into Θ_{j+1} (since a rule has been applied to Θ_j w.r.t. some other constituent of it),

(b) the occurrence of $\sigma \vdash$ in Θ_{j+1} is due to an application of a rule to Θ_j w.r.t. a constituent, say, $\theta \vdash$, of Θ_j.

If (a) is the case, then (★) holds trivially. So assume that (b) holds. Two sub-cases are possible:

(b_1) a rule has been applied to Θ_j w.r.t. the constituent $\theta \vdash$ and a term of θ annotated with k,

(b_2) a rule has been applied to Θ_j w.r.t. the constituent $\theta \vdash$ and a term of θ which is annotated with some numeral j different from k.

If (b_1) holds, then $\mathsf{wff}(f_{\setminus[k]}(\theta) \vdash) = \mathsf{wff}(f_{\setminus[k]}(\sigma) \vdash)$, so the set $\mathsf{wff}(f_{\setminus[k]}(\theta) \vdash)$ is consistent. Assume that (b_2) is the case. Suppose that the set $\mathsf{wff}(f_{\setminus[k]}(\theta) \vdash)$ is inconsistent though $\mathsf{wff}(f_{\setminus[k]}(\sigma) \vdash)$ is a consistent set. Both $\mathsf{wff}(f_{\setminus[k]}(\sigma) \vdash)$ and $\mathsf{wff}(f_{\setminus[k]}(\theta) \vdash)$ do not contain wffs annotated with k. So the hypothetical inconsistency of the set $\mathsf{wff}(f_{\setminus[k]}(\theta) \vdash)$ is due to the occurrence in θ of some wffs(s) annotated with numeral(s) different from k. Observe that the inconsistency of $\mathsf{wff}(f_{\setminus[k]}(\theta) \vdash)$ yields the inconsistency of the set $\mathsf{wff}(\theta \vdash)$. However, $\sigma \vdash$ is a constituent of Θ_{j+1} because a rule has been applied to Θ_j w.r.t. $\theta \vdash$ and a wff annotated with a numeral different from k. Thus, by Lemma 2, the set $\mathsf{wff}(\sigma \vdash)$ is inconsistent. Moreover, its inconsistency is due to the occurrence in σ of wffs annotated with numerals different from k. Therefore the set $\mathsf{wff}(f_{\setminus[k]}(\sigma) \vdash)$ is inconsistent. We arrive at a contradiction. This completes the proof of ($\bigstar$).

The sequence (45) is supposed to be a proof, so, by Definition 7, for any $k \in \{1, \ldots, m\}$ there exists a constituent, say, $\rho \vdash$, of Θ_n such that, as $f_{\setminus[k]}(\rho \vdash)$ is either $\emptyset \vdash$ or is an open atomic sequent, the set $\mathsf{wff}(f_{\setminus[k]}(\rho) \vdash)$ is consistent. Thus, by ($\bigstar$) proven above, any term/seqsequent of (45) has a constituent, $\zeta \vdash$, such that $\mathsf{wff}(f_{\setminus[k]}(\zeta) \vdash)$ is a consistent set of wffs. But the sequent $C_{\vec{m}}^{[\vec{m}]} \vdash$ is the only constituent of Θ_1. Hence $\mathsf{wff}(f_{\setminus[k]}(C_{\vec{m}}^{[\vec{m}]} \vdash)$ is a consistent set. As k was an arbitrary element of $\{1, \ldots, m\}$, by Corollary 41 it follows that X is a MI-set. □

An auxiliary concept is needed.

Definition 13 ($\mathsf{MI}^{\mathsf{CPL}}$-transformation of a sequent)**.** *A $\mathsf{MI}^{\mathsf{CPL}}$-transformation of a sequent $\sigma \vdash$ is a finite sequence $\Theta_1, \ldots, \Theta_n$ of seqsequents such that: (a) $\Theta_1 = \langle \sigma \vdash \rangle$, and (b) Θ_{j+1} results from Θ_j by a rule of $\mathsf{MI}^{\mathsf{CPL}}$ for $1 \leqslant j < n$.*

Theorem 13 (Completeness w.r.t MI-sets)**.** *If X is a MI-set, then any ordered sequent $\sigma \vdash$ such that $\mathsf{wff}(\sigma \vdash) = X$ is provable in $\mathsf{MI}^{\mathsf{CPL}}$.*

Proof. A moment's reflection on the rules of $\mathsf{MI}^{\mathsf{CPL}}$ reveals that for each ordered sequent $\sigma \vdash$ such that $\mathsf{wff}(\sigma \vdash)$ is an inconsistent set of wffs, there exist $\mathsf{MI}^{\mathsf{CPL}}$-transformations of the sequent which end with seqsequents whose constituents are closed atomic sequents only.

Assume that X is a MI-set, and that $C_{\vec{m}}^{[\vec{m}]} \vdash$ is an ordered sequent such that $\mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash) = X$. Since X is a MI-set, $\mathsf{wff}(C_{\vec{m}}^{[\vec{m}]} \vdash)$ is an inconsistent set. Let

$$\Theta_1, \ldots, \Theta_n \tag{46}$$

be a $\mathsf{MI}^{\mathsf{CPL}}$-transformation of the sequent $C_{\vec{m}}^{[\vec{m}]} \vdash$ such that each constituent of Θ_n is a closed atomic sequent. Suppose that the transformation (46) is not a proof of $C_{\vec{m}}^{[\vec{m}]} \vdash$.

The transformation (46) can be depicted as:

$$
\begin{aligned}
&\Theta_1 \hookleftarrow \mathsf{R}_x^{[i_1]} \\
&\Theta_2 \hookleftarrow \mathsf{R}_x^{[i_2]} \\
&\qquad \cdots \\
&\Theta_{n-1} \hookleftarrow \mathsf{R}_x^{[i_{n-1}]} \\
&\Theta_n
\end{aligned} \tag{47}
$$

where '$\hookleftarrow \mathsf{R}_x^{[i_j]}$' indicates that the rule applied to Θ_j acts upon a wff annotated with i_j (more precisely, upon an occurrence of such a wff in a sequent that belongs to Θ_j).

If the transformation (46) is not a proof, then, by Definition 7, there exists an index k, where $1 \leqslant k \leqslant m$, such that for each sequent $\theta \vdash$ which occurs in Θ_n, $f_{\setminus[k]}(\theta) \vdash$ is a closed atomic sequent.

Suppose that $m = 1$. Thus X is a singleton set, each rule of (47) acts upon a wff annotated with 1, and all the wffs which occur in Θ_n are annotated with 1. Hence $f_{\setminus[1]}(\theta) \vdash = \emptyset \vdash$ for any constituent $\theta \vdash$ of Θ_n. It follows that there is no constituent of Θ_n such that $f_{\setminus[1]}(\theta \vdash)$ is a closed atomic sequent. We arrive at a contradiction. Thus $m \neq 1$.

We proceed as follows. First, we remove from (47) each Θ_j which is associated with $\hookleftarrow \mathsf{R}_x^{[k]}$, that is, we skip all the lines of (47) in which a rule acts upon a wff annotated with k. Let

$$\Theta_1^*, \ldots, \Theta_h^* \tag{48}$$

stand for the subsequence of (46) obtained from it in this way. Each Θ_j^*, where $1 \leqslant j \leqslant h$, is a sequence of sequents. Let $\Theta_j^* = \langle \xi_1 \vdash, \ldots, \xi_s \vdash \rangle$. We define Θ_j^{**} as:

$$\langle f_{\setminus[k]}(\xi_1) \vdash, \ldots, f_{\setminus[k]}(\xi_s) \vdash \rangle \tag{49}$$

Since $f_{\setminus[k]}(\xi_i) \vdash$ is a sequent for $1 \leqslant i \leqslant s$, (49) is a seqsequent. Then we consider the following sequence of seqsequents:

$$\Theta_1^{**}, \ldots, \Theta_h^{**} \tag{50}$$

Observe that wffs annotated with k do not occur in any constituent of any element/term of (50). Clearly, $f_{\setminus[k]}(C_{\overrightarrow{m}}^{[\overrightarrow{m}]}) \vdash$ is

$$C_1^{[1]}, \ldots, C_{k-1}^{[k-1]}, C_{k+1}^{[k+1]}, \ldots, C_m^{[m]} \vdash \tag{51}$$

It is easily seen that (50) is a $\mathsf{MI}^{\mathsf{CPL}}$-transformation of the sequent (51). On the other hand, each constituent of Θ_h^{**} is a closed atomic sequent and hence $\mathsf{wff}(\theta \vdash)$ is inconsistent for any constituent $\theta \vdash$ of Θ_h^{**}. Thus, by Lemma 4,

$$\{C_1, \ldots, C_{k-1}, C_{k+1}, \ldots, C_m\} \tag{52}$$

is an inconsistent set of wffs. But (52) is a proper subset of X. Therefore X is not a MI-set. We arrive at a contradiction. This completes the proof. □

Theorem 14. *If there exists a* $\mathsf{MI}^{\mathsf{CPL}}$*-disproof of an ordered sequent* $C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash$*, then the set* $\mathsf{wff}(C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash)$ *is inconsistent, but is not a* MI*-set.*

Proof. Let

$$\Theta_1', \ldots, \Theta_n' \tag{53}$$

be an arbitrary but fixed disproof of $C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash$. Everything what has been said, in the above proof of Theorem 13, about the transformation (46), can be repeated with regard to the disproof (53) (of course, after replacing Θ_j with Θ_j' for $1 \leqslant j \leqslant n$). So $\mathsf{wff}(C_{\overrightarrow{m}}^{[\overrightarrow{m}]} \vdash)$ is not a MI-set. Yet, due to Definition 8 and Lemma 3, it is an inconsistent set. □

Theorem 15. *If* X *is a finite inconsistent set of wffs which is not a* MI*-set, then any ordered sequent* $\sigma \vdash$ *such that* $\mathsf{wff}(\sigma \vdash) = X$ *is disprovable in* $\mathsf{MI}^{\mathsf{CPL}}$*.*

Proof. Let X be an arbitrary but fixed finite inconsistent set of wffs which is not a MI-set. We define:

$$\Sigma = \{\sigma \vdash : \mathsf{wff}(\sigma \vdash) = X \textit{ and } \sigma \vdash \textit{ is an ordered sequent}\}$$

Let Λ be the set of all $\mathsf{MI}^{\mathsf{CPL}}$-transformations of sequents in Σ. As X is inconsistent, Λ includes a non-empty subset Λ_0 of $\mathsf{MI}^{\mathsf{CPL}}$-transformations each of which ends with a seqsequent involving closed atomic sequent(s) only. By assumption, X is not a MI-set. Thus, by Theorem 12, no element of Λ_0 has a $\mathsf{MI}^{\mathsf{CPL}}$-proof. Therefore each transformation in Λ_0 violates the fourth clause of Definition 7. Hence Λ_0 comprises $\mathsf{MI}^{\mathsf{CPL}}$-disproofs of the sequents in Σ. □

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